Alternative axioms related to Playfair's Postulate.

Definition 1. Let Π be a Hilbert plane (meaning a plane satisfying I1-3, B1-4, C1-6). We call Π

1. *semi-Euclidean* if triangle sums are $= 2$RA,
2. *semi-hyperbolic* if triangle sums are < 2RA,
3. *semi-elliptic* if triangle sums are > 2RA.

You will find in Theorem 34.7 a proof that if one triangle has angle sum =2RA ($<$ 2RA, $>$ 2RA), then all do. Computing an angle sum is a local way to determine approximately how close Euclidean the plane is.

The *defect* of triangle $\triangle ABC$ is 2RA - (angle sum). Thus semi-Euclidean planes have triangles of defect 0, while semi-hyperbolic planes have triangles of positive defect.

Alternative axioms related to (P):

1. (Archimedes Axiom) If $\overline{AB}$ and $\overline{CD}$ are segments, then there is an integer n such that $n \cdot \overline{AB} > \overline{CD}$.
2. (Aristotle's Axiom) If $\angle ABC$ is an angle and $\overline{DE}$ is a segment, then there is a point F on $\overrightarrow{AB}$ such that, if G is the foot of the projection of F onto AC, we have $\overline{FG} > \overline{DE}$.
3. (Clavius's Axiom) Given line ℓ and point A not on ℓ, the equidistant locus to ℓ through P is a line.
4. (Wallis's Axiom) Given triangle $\triangle ABC$ and segment $A'B'$, there is a C' such that corresponding angles in the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.
5. (Dedekind's Completeness Axiom) Every "Dedekind cut" is "defined by a point". Here a Dedekind cut is a partition $\{X, Y\}$ of the points incident to some line into two nonempty convex subsets. A Dedekind cut $\{X, Y\}$ is determined by a point A if one of the sets X, Y is the set of all points on ℓ on one side of A, and the other set is A together with all points on the other side. (See page 115.)
6. (Existence of Limiting Parallels) Let ℓ be a line, A a point not on ℓ, and m be the line through A that meets ℓ at B in a right angle. A limiting parallel (page 312) is a ray $\overrightarrow{AC}$ which does not meet ℓ, but any ray $\overrightarrow{AD}$ with D interior to $\angle BAC$ meets ℓ.

Some Examples:

1. The Cartesian plane over $\mathbb{R}$ is semi-Euclidean and satisfies all of these axioms.
2. The Dehn plane (= the Cartesian plane over the ring of bounded elements of a non-Archimedean ordered field) is semi-Euclidean and satisfies Clavius's Axiom, but fails all of the rest of these axioms.

Some Relationships:

1. A Hilbert plane satisfies Clavius's Axiom iff it is semi-Euclidean iff rectangles exist.
2. A Hilbert plane satisfies Playfair's Postulate iff it satisfies Wallis's Axiom iff it is Archimedean and semi-Euclidean.
3. Dedekind's Axiom implies the existence of limiting parallels.
4. The Cartesian plane over a Euclidean field satisfies Archimedes Axiom iff the field is isomorphic to a subfield of $\mathbb{R}$. The plane satisfies Dedekind's Axiom iff the field is isomorphic to $\mathbb{R}$ itself.