Logic.

The fact that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function is sometimes written

$$\forall x_0\ \forall \varepsilon\ \exists \delta\ \forall x\ ((\varepsilon \le 0) \vee ((0 < |x - x_0| < \delta) \to (|f(x) - f(x_0)| < \varepsilon)))$$

We will learn how to read and write such sentences, and how to test whether a sentence like this is true.

We will discuss the following four topics.

- **Syntax.** (What are the accepted ways to combine symbols into sensible expressions?)
- **Meaning.** (What does it mean for an expression to be true or false?)
- **Proof.** (How can we determine truth or communicate it to others?)
- **The relationship between truth and provability.** (Are all provable statements true? Are all true statements provable?)

(I) Syntax.
  (1) An alphabet of symbols.
    (a) variables: $x_1, x_2, x_3, \ldots$
    (b) logical symbols
      (i) equality: $=$
      (ii) logical connectives: $\wedge, \vee, \neg, \to, \leftrightarrow$.
      (iii) quantifiers: $\exists, \forall$
    (c) nonlogical symbols: operation symbols, predicate symbols
    (d) punctuation: parentheses, commas.
  (2) Formulas and sentences.
(II) Meaning.
  (1) The logical connectives.
  (2) The equivalence of $(H \to C)$, $((\neg C) \to (\neg H))$ and $((H \wedge \neg C) \to \text{False})$.
  (3) Structures.
  (4) Valuations.
  (5) The symbol $\models$
  (6) The equivalence of $\neg(\exists x P)$ and $\forall x(\neg P)$.
(III) Proof.
  (1) Axioms.
  (2) Rules of deduction.
  (3) The symbol $\vdash$
(IV) The relationship between truth and provability.

Practice Problems.

Write the following formally (in the prescribed language).

(1) Every positive number has a square root. Use the language of fields with a special symbol for square root. (This is the language of $\langle \mathbb{R}; +, -, 0, \cdot, 1, \sqrt{\ }\rangle$.)

(2) Every positive number has a square root. Use the language of fields. (This is the language of the structure $\langle \mathbb{R}; +, -, 0, \cdot, 1\rangle$. You have no symbol for square root.)

(3) There exist four points with no three collinear. Use the language of incidence geometry.

(4) In 1959, Pete Seeger took lines from the Book of Ecclesiastes to write a song, which was made famous by the Byrds in 1965. One line is:

To every thing there is a season, and a time for every purpose under heaven.

Write this as a formal sentence using predicates $S(s,t) =$ "s is the season for thing t" and $P(T,p) =$ "T is the time for purpose p". (The quantifier order is tricky in this example!)

(5) Write a formal sentence that expresses

You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

Use a predicate $Fool(p,t)$, which is defined to hold if person p can be fooled at time t.