Euclidean and Non-Euclidean Geometry
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define the following terms. Remember to indicate what kind of mathematical object is being defined. (Is it a set? a number? a function?)

(a) isomorphism of incidence geometries

An isomorphism from $\pi = \langle \mathcal{P}, \mathcal{L}; I\rangle$ to $\pi' = \langle \mathcal{P}', \mathcal{L}'; I'\rangle$ is a pair of functions $F = (F_P, F_L)$, with $F_P\colon \mathcal{P} \to \mathcal{P}'$ and $F_L\colon \mathcal{L} \to \mathcal{L}'$, such that (i) F_P and F_L are bijections, and (ii) F_P and F_L preserve and reflect incidence in the sense that

$$A \; I \; \ell \;\longleftrightarrow\; F_P(A) \; I' \; F_L(\ell).$$

A less formal way to say this is that an isomorphism is a pair of functions $F = (F_P, F_L)$ such that F_P and F_L are bijections between point sets and line sets respectively, which together induce a bijection between incidence sets.

(b) triangle

Give noncollinear points A, B and C, the triangle they determine, ABC, is the set $\overline{AB} \cup \overline{BC} \cup \overline{CA}$.

(c) ray

Given distinct points A, B, the ray $\overrightarrow{AB}$ is the set of points that includes point A and all points incident to line AB which are on the same side as B.

An equivalent way to say this is that $\overrightarrow{AB}$ is the set

$$\{A, B\} \cup \{C \mid A * C * B\} \cup \{C \mid A * B * C\}$$

(or $\overline{AB} \cup \{C \mid A * B * C\}$, which is the same set).

2. Explain why $\{B1\} \not\models B2$.

We must produce a model of $\{B1, \neg B2\}$. This will be a structure satisfying $B1$, but not $B2$. To fail $B2$ our model must have at least two points, but we really don't need any other points. We don't even need lines! If we choose the model to have 2 points, 0 lines, and 0 betweenness triples, then $B1$ will be satisfied while $B2$ will not be satisfied.

3. Write Betweenness Axiom 1 as a formal sentence.

To fit the sentence on one line, let me use abbreviations. Let $\text{distinct}(A, B, C)$ abbreviate the distinctness relation for three points:

$$(\neg(A = B)) \wedge (\neg(B = C)) \wedge (\neg(A = C)).$$

Let $\text{collinear}(A, B, C)$ abbreviate the collinearity relation:

$$\exists \ell (AI\ell \wedge BI\ell \wedge CI\ell).$$

The sentence we need is

$$\forall A \forall B \forall C((A * B * C) \to \text{distinct}(A, B, C) \wedge \text{collinear}(A, B, C) \wedge (C * B * A)).$$

4. Let ABC be a triangle. Choose points D, E and F, one from each side of ABC and none equal to A, B or C. Explain why DEF is also a triangle.

Two sides of a triangle meet only at a vertex, and D, E and F are not vertices of ABC, so D, E, F must be distinct points.

If DEF is not a triangle, then D, E, F must be collinear; let ℓ be the line incident to all three. Now (i) ℓ cannot pass through any vertex of ABC, and (ii) ℓ enters the triangle at (say) D, so by Pasch's Axiom ℓ must exit through exactly one other side. But ℓ is assumed to exit through both E and F, which are on different sides of ABC. This contradiction completes the argument.