How to construct a Pythagorean ordered field from a model of neutral geometry satisfying Playfair's Axiom.

(I) Choose a line, x. (I use x because I am thinking of it as the x-axis of $\mathbb{R}^2$.) Let F be the set of points incident to x. These will be the elements of our field.

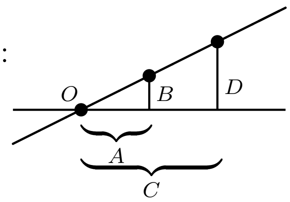
(II) Choose distinct points $O, I \in F$. The point O will play the role of zero in our field, and the point I will play the role of 1. Let P be the set of points on x on the same side of O as I. These are meant to be the positive numbers. One can also think of the elements of P as representing congruence classes of segments, or even as representing nonzero segment lengths.

(III) If $A \in P$, then there is a unique element A' on x on the other side of O such that $\overline{OA} \cong \overline{OA'}$. Write $-A$ for A'. Let $-O = O$. This defines the negation operation on F.

(IV) Define $+$ on P by using segment addition. Extend this definition to all of F so that $O + A = A + O = A$, $(-A) + (-B) = -(A + B)$, and $A + (-B) = (-B) + A = C$ iff $A = B + C$.

(V) Define $\leq$ on P using betweenness. That is, $A \leq B$ iff $A = B$ or $O * A * B$. Then extend the definition to all of F by stating that for $A, B \in P$, (i) $-A \leq -B$ iff $B \leq A$, (ii) $-A \leq B$, (iii) $-A \leq 0 \leq A$.

(VI) Define proportions $\frac{A}{B} = \frac{C}{D}$ of elements $A, B, C, D \in P$:



(VII) Use proportion to define multiplication and inverses of positive elements. Extend the definitions to all of F.

(VIII) Show that $\mathbb{F} = \langle F; +, -, O, \cdot, I, \leq \rangle$ satisfies the axioms of ordered fields.

(IX) Define two triangles to be similar if corresponding angles are congruent and corresponding sides have the same proportion. (Section 20.) Show (Proposition 20.1, AAA) that if corresponding angles are congruent, then corresponding sides must have the same proportion, so you can drop the second condition from the definition.

(X) Show that the sum of three angles of a triangle equals a straight line (more carefully: the sum of two angles is congruent to the supplement of the third). In particular, show that, given two triangles, if two corresponding angles are congruent, then the triangles are similar.

(XI) Prove the Pythagorean Theorem.

(XII) Show that the field associated to a model of neutral geometry satisfying Playfair's Axiom must be clased under the Pythagorean operation $p(x, y) = \sqrt{x^2 + y^2}$.