The Hilbert Field, part 2.

In order to understand the geometry of the Cartesian plane over Ω, we need to know something about which numbers lie in Ω. Since Ω is the smallest subfield of $\mathbb{C}$ that is closed under $0, 1, x+y, -x, x\cdot y, \frac{1}{x}, \sqrt{x^2+y^2}$, one can show that a number $z \in \mathbb{C}$ belongs to Ω by producing a *construction sequence.* For example, to show $\sqrt{2+\sqrt{2}} \in \Omega$ we write down

$$\left(0, 1, \sqrt{2}, \frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, \sqrt{2+\sqrt{2}}\right).$$

This shows that $\sqrt{2+\sqrt{2}} \in \Omega$, since

(1) $0, 1 \in \Omega$, since they belong to any subfield of $\mathbb{C}$.
(2) $\sqrt{2}$ is constructed from earlier numbers in the sequence using the Pythagorean operation: $\sqrt{2} = \sqrt{1^2+1^2}$.
(3) $\frac{1}{\sqrt{2}}$ is the inverse of an earlier-constructed number.
(4) $1+\frac{1}{\sqrt{2}}$ is the sum of two earlier-constructed numbers.
(5) $\sqrt{2+\sqrt{2}}$ is constructed from earlier numbers using the Pythagorean operation:

$$\sqrt{2+\sqrt{2}} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1+\frac{1}{\sqrt{2}}\right)^2}.$$

It is important to observe that the set S of all complex numbers that have a construction sequence (i) is contained in Ω, and (ii) is closed under $0, 1, x+y, -x, x\cdot y, \frac{1}{x}, \sqrt{x^2+y^2}$, and (using the order of the real numbers) is a Pythagorean order field, hence contains Ω. Together these observations show that $\Omega = S$. In other words, Ω is exactly the set of complex numbers that have a construction sequence.

The number $\sqrt{1+\sqrt{2}}$ looks very much like $\sqrt{2+\sqrt{2}}$, but strangely does not belong to Ω. This is harder to show. We will show it by arguing that

(A) All elements of Ω are totally real.
(B) $\sqrt{1+\sqrt{2}}$ is not totally real.

Thm A. All elements of Ω are totally real.

Idea of Proof. Show that every element with a construction sequence is totally real. To do this, first show that $0, 1$ are totally real. Then show that if x, y are totally real, then $x+y, -x, x\cdot y, \frac{1}{x}$, and $\sqrt{x^2+y^2}$ are also totally real. (To do this correctly, one should prove the theorem by induction on the length of a construction sequence, and for this it helps to allow both $+\sqrt{x^2+y^2}$ and $-\sqrt{x^2+y^2}$ as single construction steps.) $\square$

Thm B. $\sqrt{1+\sqrt{2}}$ is not totally real.

Idea of Proof. There is an automorphism $\alpha\colon \mathbb{C} \to \mathbb{C}$ such that $\alpha(\sqrt{2}) = -\sqrt{2}$. For this automorphism we have $\alpha(\sqrt{1+\sqrt{2}}) = \pm\sqrt{\alpha(1)+\alpha(\sqrt{2})} = \pm\sqrt{1-\sqrt{2}}$, which is not real. $\square$