**Ordered Fields.**

**Definition 1.** A **field** is an algebraic structure $\mathbb{F} = \langle F; +, -, 0, \cdot, 1\rangle$ which satisfies the following

(1) Additive laws[1]:
   (a) (Associative law) $\forall x \forall y \forall z((x + (y + z)) = ((x + y) + z))$.
   (b) (Commutative law) $\forall x \forall y(x + y = y + x)$.
   (c) (Unit law) $\forall x(x + 0 = x)$
   (d) (Inverse law) $\forall x(x + (-x) = 0)$
(2) Multiplicative laws:
   (a) (Associative law) $\forall x \forall y \forall z((x(yz)) = ((xy)z))$.
   (b) (Commutative law) $\forall x \forall y(xy = yx)$.
   (c) (Unit law) $\forall x(x1 = x)$
(3) Law linking addition to multiplication:
   (a) (Distributive law): $\forall x \forall y \forall z(x(y + z) = xy + xz)$.
(4) Other defining properties that are not laws:
   (a) $0 \neq 1$.
   (b) $\forall x((\neq (x = 0)) \to \exists y(xy = 1))$.

If you know the definition of "abelian group", the axioms for fields say that $\mathbb{F}$ is additively an abelian group, $\mathbb{F} - \{0\}$ is multiplicatively an abelian group, and the additive and multiplicative structures are linked by the distributive law.

Examples. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}$ ($p$ prime).
Nonexamples. $\mathbb{N}, \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$ ($n$ not a prime).

**Definition 2.** An **ordered field** is a structure $\mathbb{F} = \langle F; +, -, 0, \cdot, 1, \leq\rangle$ where

(1) $\langle F; +, -, 0, \cdot, 1\rangle$ is a field,
(2) $\langle F; \leq\rangle$ is a totally ordered set (which means that $\leq$ is a reflexive, antisymmetric, transitive relation satisfying the trichotomy law),
(3) (Order structure is linked to field structure):
   (a) (Additive compatibility) $\forall x \forall y \forall z((y \leq z) \to (x + y \leq x + z))$
   (b) (Multiplicative compatibility) $\forall x \forall y \forall z(((y \leq z) \wedge (0 \leq x)) \to (xy \leq xz))$

If $\mathbb{F}$ is an ordered field, then an element $p$ is **positive** if $0 \leq p$ and $0 \neq p$ (we just abbreviate this with $0 < p$). The set $P$ of all positive elements is called the **positive cone**. An element in $-P = \{-x \mid x \in P\}$ is called **negative**.

Examples. $\mathbb{Q}, \mathbb{R}$.
Nonexamples. $\mathbb{C}, \mathbb{Z}/n\mathbb{Z}$ for any $n$.

[1] A **law** or **identity** is a universally quantified equation.

Question: Why is impossible to order $\mathbb{C}$ to make it an ordered field?

**Thm.** If $\mathbb{F}$ is an order field with positive cone $P$, then the following are true.

(1) (Can use $<$ in place of $\leq$)
  (a) $\forall x \forall y \forall z((y < z) \to (x + y < x + z))$
  (b) $\forall x \forall y \forall z(((y < z) \wedge (0 < x)) \to (xy < xz))$

(2) $\mathbb{F}$ is the disjoint union $P \cup \{0\} \cup -P$.

(3) The positive cone of $\mathbb{F}$ is closed under addition and multiplication.

(4) For any $a \in F - \{0\}$, $a^2$ is positive.

*Proof.*
I prove (1)(a) and leave (1)(b) as an exercise. Choose arbitrary field elements $a, b, c$ and assume that $b < c$. We want to prove that $a + b < a + c$. Since $b < c$, we have $b \leq c$, so $a + b \leq a + c$ by (3)(a) of the definition of ordered fields. To show that $a + b < a + c$ holds we must rule out the possibility that $a + b = a + c$. For the purpose of obtaining a contradiction, assume that we do have $a + b = a + c$. Then using the field laws we deduce in order that

$$\begin{aligned}
(-a) + (a + b) &= (-a) + (a + c) \\
((-a) + a) + b) &= ((-a) + a) + c), \\
((a + (-a)) + b) &= ((a + (-a)) + c), \\
(0 + b) &= (0 + c), \\
(b + 0) &= (c + 0), \\
b &= c,
\end{aligned}$$

which is false. This contradiction shows that $\neg(a + b = a + c)$, so with our earlier conclusion (that $a + b \leq a + c$) we get $a + b < a + c$.

Item (2) follows from the law of trichotomy. (For every $x$, exactly one of the following is true: $0 < x$, $0 = x$, or $x < 0$. In the first case, $x \in P$; in the second, $x = \{0\}$; in the third, add $-x$ ro both sides of $x < 0$ to obtain $0 < -x$, so $-x \in P$, so $x \in -P$.)

Item (3) follows from Item (1) by letting $y$ be 0 and $x$ be positive.

For Item (4), choose any $a \in F - \{0\}$. By Item (2), either $a \in P$ or $a \in -P$ (equivalently $-a \in P$). Therefore one of $a^2$ or the equal value $(-a)^2$ is the square of a positive element. By Item (3) we get $a^2 = (-a)^2 \in P$. □

**Cor.** If $\mathbb{F}$ is a field in which $-1$ is a sum of squares, then $\mathbb{F}$ is not orderable.

*Proof.* Any nonzero sum of squares, such as $1 = 1^2$, belongs to $P$. Hence $-1 \in -P$. Since $P$ contains all sums of squares, $-P$ contains $-1$, and $P \cap -P = \emptyset$, we get that $-1$ cannot be a sum of squares. □

**Cor.** The fields $\mathbb{C}$ and $\mathbb{Z}/p\mathbb{Z}$ are not orderable.

*Proof.* In these fields, $-1$ is a sum of squares.
For $\mathbb{C}$, $-1$ is already a square.
For $\mathbb{Z}/p\mathbb{Z}$, 1 is a square, so $1 + 1 = 2$ is a sum of squares, and $1 + 1 + 1 = 3$ is a sum of squares, ETC. Thus every element of $\mathbb{Z}/p\mathbb{Z}$ is a sum of squares. □