The Field of Real Constructible Numbers.

The field K of real constructible numbers is the smallest Euclidean subfield of \mathbb{C} , which means that it is the smallest subfield of \mathbb{C} closed under square roots of positive real numbers. To show that a number $z \in \mathbb{C}$ belongs to K, write down a construction sequence. To show that $z \notin K$, one can use the following theorem.

Theorem 1. A number $z \in K$ has the following properties.

- (1) z is real.
- (2) z is algebraic. (This means that z is a root of a nonzero rational polynomial.)
- (3) The minimal polynomial of z over \mathbb{Q} , $\min_{z,\mathbb{Q}}(x)$ has degree that is a power of 2.
- (4) The size of the Galois group of $\min_{z,\mathbb{Q}}(x)$ is a power of 2.
- (5) Any subcollection $H \subseteq \text{Gal}(\min_{z,\mathbb{Q}}(x))$ that is closed under multiplication has size that is a power of 2.

Arithmetic Consequences.

- (1) $\sqrt{\pi} \notin K$. ($\sqrt{\pi}$ is not algebraic.)
- (2) $\sqrt[3]{2} \notin K$. $(\min_{\sqrt[3]{2},\mathbb{Q}}(x) = x^3 2$ has degree 3, which is not a power of 2.)
- (3) $\cos(2\pi/n) \in K$ iff $\phi(n)$ is a power of 2.

Geometric Consequences.

- (1) It is impossible to square the circle with straightedge and compass.
- (2) It is impossible to duplicate the cube with straightedge and compass.
- (3) An angle of the form $2\pi/n$ is constructible with straightedge and compass iff $\phi(n)$ is a power of 2. In particular, $2\pi/3$ is constructible, while $2\pi/9$ is not.