(Non-)Archimedean fields.

Let $\mathbb{F}$ be an ordered field. For $a, b \in \mathbb{F}$, the *interval* $[a, b]$ is the set $\{x \in \mathbb{F} \mid a \leq x \leq b\}$.

Theorem 1. *The following are equivalent for an ordered field* $\mathbb{F}$*.*

(1) $F = \bigcup_{n=1}^{\infty}[-n, n]$
(2) *There is no element* $t \in F$ *such that* $n < t$ *for every positive integer* n*.*
(3) *There is no* $u \in F$ *such that* $0 < u < 1/n$ *holds for every positive integer* n*.* □

$\mathbb{F}$ is *Archimedean* is it satisfies these properties, otherwise it is *non-Archimedean.* An element t satisfying the condition in Item (2) is called an *infinitely large element.* An element u satisfying the condition in Item (2) is called an *infinitely small element.*

Any ordered subfield of the real numbers, like $\mathbb{Q}, \Omega, K, \mathbb{R}$, is Archimedean.

Non-Archimedean fields exist.

Let $\mathbb{F}$ be any ordered field. The set of rational functions over $\mathbb{F}$,

$$\mathbb{F}(t) = \left\{ \frac{p(t)}{q(t)} \;\middle|\; p, q \text{ are polynomials over } \mathbb{F} \text{ in the variable } t,\ q \neq 0 \right\}$$

can be made into a non-Archimedean ordered field. By adjusting the signs in the numerator and denominator of a fraction we can write a typical element of $\mathbb{F}(t)$

$$\frac{p(t)}{q(t)} = \frac{a_m t^m + \cdots + a_1 t + a_0}{b_n t^n + \cdots + b_1 t + b_0}$$

with $b_n > 0$.

(1) $0(t)$ is the zero function.
(2) $1(t)$ is the conszero function.ant function with value 1.
(3) $\frac{p(t)}{q(t)} + \frac{r(t)}{s(t)} = \frac{p(t)s(t)+q(t)r(t)}{q(t)s(t)}$.
(4) $-\frac{p(t)}{q(t)} = \frac{-p(t)}{q(t)}$.
(5) $\frac{p(t)}{q(t)} \cdot \frac{r(t)}{s(t)} = \frac{p(t)r(t)}{q(t)s(t)}$.
(6) If $\frac{p(t)}{q(t)} \neq 0(t)$ (so $p(t) \neq 0(t)$), then $\left(\frac{p(t)}{q(t)}\right)^{-1} = \frac{q(t)}{p(t)}$. (Might have to adjust the sign of the denominator.)
(7) $\frac{p(t)}{q(t)}$ is positive if the leading coefficient of $p(t)$ is positive.
(8) $\frac{r(t)}{s(t)} < \frac{r(t)}{s(t)}$ iff $\frac{r(t)}{s(t)} - \frac{p(t)}{q(t)}$ is positive.

$\mathbb{F}(t)$, with this ordering, is an ordered field. The element t is infinitely large in $\mathbb{F}(t)$. The element $1/t$ is infinitely small but positive in $\mathbb{F}(t)$.