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Congruence join semidistributivity is equivalent to a congruence identity

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Abstract. We show that a locally finite variety is congruence join semidistributive if and only if it satisfies a congruence identity that is strong enough to force join semidistributivity in any lattice.

1. Introduction

The *join semidistributive law* for lattices is the implication

$$\alpha \vee \beta = \alpha \vee \gamma \implies \alpha \vee \beta = \alpha \vee (\beta \wedge \gamma).$$

Recursively define lattice words in the variables α, β and γ by $\beta^0 = \beta$, $\gamma^0 = \gamma$, $\beta^{n+1} = \beta \wedge (\alpha \vee \gamma^n)$, and $\gamma^{n+1} = \gamma \wedge (\alpha \vee \beta^n)$. A result of Jónsson and Rival in [6] implies that any finite lattice satisfying the join semidistributive law satisfies one of the weakened distributive laws:

$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta^n) \wedge (\alpha \vee \gamma^n), \qquad E^n:$$

and that any lattice that satisfies some E^n satisfies the join semidistributive law. In this paper we show that if $\mathcal{V}$ is a locally finite variety, then the congruence lattices of members of $\mathcal{V}$ are congruence join semidistributive if and only if these congruence lattices satisfy E^n for some fixed n.

Join semidistributivity and the dual property, meet semidistributivity, play important roles in the classification of varieties according to congruence properties. This is best understood from the viewpoint of commutator theory. In the nicest situations, if one restricts an algebra's operations to a block of an abelian congruence, then those operations have a representation that is linear with respect to some abelian group. The commutator which is used to define this concept of abelianness is linked to congruence meet semidistributivity by the fact proved in [11] that a variety omits abelian congruences if and only if the variety is congruence meet

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semidistributive. A different concept of "abelianness", called rectangularity, is analyzed in [9]. If one restricts an algebra's operations to a block of a self-rectangulating tolerance, then those operations have a representation that is linear with respect to some semilattice. It is proved in [9] that a variety omits self-rectangulating tolerances if and only if the variety satisfies a nontrivial congruence identity. It is also shown that congruence join semidistributivity is strong enough to force the omission of abelian congruences and self-rectangulating tolerances. Thus, it is plausible that a variety is congruence join semidistributive precisely when its members have no abelian congruences and no self-rectangulating tolerances. In other words, it is plausible that a variety is congruence join semidistributive if and only if it is congruence meet semidistributive and satisfies a nontrivial congruence identity. In this paper we prove this statement for locally finite varieties. We also resolve the following questions about locally finite varieties:

(1) (D. Hobby and R. McKenzie, page 143 of [5]) If the finite algebras in a locally finite variety are congruence join semidistributive, must all algebras be congruence join semidistributive? (Answer: Yes.)
(2) (G. Czédli, [2]) Can congruence join semidistributivity be characterized by a Mal'tsev condition? (Answer: Yes, for locally finite varieties.)

In addition to these new results, Theorem 2.4 augments earlier results in [7] about congruence identities in locally finite varieties.

2. Bounding the length of a herringbone

Let **L** be a lattice. By a *herringbone* in **L** we mean a subset of three descending chains $\{\alpha^i\}$, $\{\beta^{2i}\}$, $\{\gamma^{2i+1}\}$ in **L** where $\{\alpha^i\} \cup \{\beta^{2i}\}$ and $\{\alpha^i\} \cup \{\gamma^{2i+1}\}$ are sublattices of **L** ordered as in Figure 1.

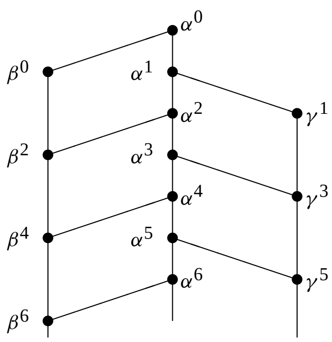


Figure 1 A Herringbone

In other words, a herringbone is a partial sublattice of **L**, ordered as in Figure 1, with

(β) $\alpha^{2i+1} \wedge \beta^{2i} = \beta^{2i+2}$, $\alpha^{2i+2} \vee \beta^{2i} = \alpha^{2i}$, and
(γ) $\alpha^{2i+2} \wedge \gamma^{2i+1} = \gamma^{2i+3}$, $\alpha^{2i+3} \vee \gamma^{2i+1} = \alpha^{2i+1}$.

If there are infinitely many distinct α's, then we will say that the *length* of the herringbone is ∞. Otherwise, the *length* of the herringbone is the supremum of the superscripts k such that $\alpha^0 > \alpha^1 > \cdots > \alpha^k$. (It is easy to see from (β) and (γ) that if some $\alpha^k = \alpha^{k+1}$ then $\alpha^k = \alpha^{k+1} = \alpha^{k+2} = \cdots$.) Our first goal is to prove that if a locally finite variety $\mathcal{V}$ omits types **1** and **5**, then there is some positive integer N which bounds the length of any herringbone that appears in the congruence lattice of a member of $\mathcal{V}$. The proof requires tame congruence theory, and the reader is referred to [5] for the details of the theory.

Then notation $a \equiv_\theta b$ will be used to mean that θ is an equivalence relation and $(a, b) \in \theta$.

Recall from [1] that if **A** is a finite algebra and δ and θ are congruences of **A** for which $\delta \prec \theta$ in Con(**A**), then a two-element set $\{0, 1\}$ is a $\langle\delta, \theta\rangle$-*subtrace* if $(0, 1) \in \theta - \delta$ and $\{0, 1\}$ is a subset of a $\langle\delta, \theta\rangle$-minimal set.

DEFINITION 2.1. (From [7].) Let **A** be a finite algebra with congruences $\delta \prec \theta$. Let $K = Int[\delta, \theta]$ be the two-element interval in Con(**A**) determined by these congruences. If f and g are terms of **A**, then we will use the notation

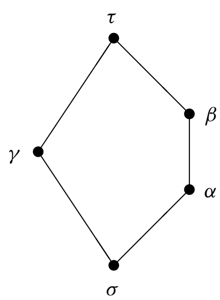
$$f(x_1, \ldots, x_n) \approx_K g(x_1, \ldots, x_n)$$

to mean that whenever $\{0, 1\}$ is a $\langle\delta, \theta\rangle$-subtrace, then $f(x_1, \ldots, x_n) \equiv_\delta g(x_1, \ldots, x_n)$ holds if all x_i belong to $\{0, 1\}$.

If $\alpha < \beta$ are congruences on **A** and $I = Int[\alpha, \beta]$, then we will write $f(\mathbf{x}) \approx_I g(\mathbf{x})$ to mean that $f(\mathbf{x}) \approx_K g(\mathbf{x})$ holds whenever K is a two-element subinterval of I. We call equations of the type $f \approx_I g$ *local equations*.

The next two lemmas have slightly different hypotheses than Lemmas 3.2 and 3.4 of [7], but the proofs are similar.

LEMMA 2.2. *Let* **A** *be a finite algebra for which* typ$\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. *Assume that* Con(**A**) *has a sublattice isomorphic to the pentagon, with congruences* $\{\sigma, \gamma, \alpha.\beta, \tau\}$ *labeled as in the following figure.*



Let $I = Int[\beta, \tau]$ *and* $J = Int[\alpha, \beta]$*. If* **A** *satisfies the local equations* $f(x, y, y) \approx_I x$ *and* $f(x, y, y) \approx_J x$*, then* **A** *satisfies both of the local equations*

$$f(x, y, x) \approx_J x \quad \text{and} \quad f(x, x, y) \approx_J x.$$

Proof. It is enough to prove that if $f(x, y, y) \approx_I x$ and $f(x, y, y) \approx_J x$ hold, then $f(x, x, y) \approx_J x$ holds. The same argument applied to $f'(x, y, z) = f(x, z, y)$ shows that $f(x, y, y) \approx_I x$ and $f(x, y, y) \approx_J x$ imply $f(x, y, x) \approx_J x$.

Assume to the contrary that $f(x, y, y) \approx_I x$ and $f(x, y, y) \approx_J x$ while $f(x, x, y) \not\approx_J x$. Since $f(x, x, y) \not\approx_J x$, there exist a two-element subinterval $K = Int[\delta, \theta]$ of J, and a $\langle\delta, \theta\rangle$-subtrace $\{0, 1\}$ such that $w = f(0, 0, 1) \not\equiv_\delta 0$. We shall derive a contradiction to this.

Assume that $\text{typ}(\delta, \theta) = \mathbf{4}$. The θ-block containing 0 and 1 is connected by a δ-closed preorder that is compatible with all operations of **A**, which has the property that distinct elements of a two-element subtrace are comparable. (See Theorem 5.26 of [5]). So, if $\{x, y\} = \{0, 1\}$, then $f(x, x, x)$ and $f(x, y, y)$ are comparable elements of this θ-block. The element $f(x, x, y)$ is in the interior of the interval determined by $f(x, x, x)$ and $f(x, y, y)$. For us, $f(x, x, x) \equiv_\delta x \equiv_\delta f(x, y, y)$, so we deduce that $f(x, x, y) \equiv_\delta x$ as well. This leads to $w = f(0, 0, 1) \equiv_\delta 0$, which is false. It must be that $\text{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}\}$.

Continue to assume that $\{0, 1\}$ is a $\langle\delta, \theta\rangle$-subtrace for which $w = f(0, 0, 1) \not\equiv_\delta 0$. Let U be a $\langle\delta, \theta\rangle$-minimal set which has a $\langle\delta, \theta\rangle$-trace N containing 0 and 1. Since $w = f(0, 0, 1) \equiv_\theta f(0, 0, 0) \equiv_\delta 0$, we have $(w, 0) \in \theta - \delta$. Since $\text{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}\}$, Lemma 4.7 of [10] applies to show that there is an idempotent unary polynomial e such that $e(A) = U$ and $(e(w), e(0)) = (e(w), 0) \notin \delta$.

Since $\alpha \leq \delta \prec \theta \leq \beta \leq \tau$, there is a chain of congruences

$$\theta = \lambda_0 \prec \lambda_1 \prec \cdots \prec \lambda_k = \tau$$

which contains β. In any such chain, each interval $K_i = Int[\lambda_i, \lambda_{i+1}]$ is contained in either I or J. Thus, the hypotheses of the lemma guarantee that we have $f(x, y, y) \approx_{K_i} x$ for each i. Let B and T denote the body and tail of U. By Claim 3.1 of [7], the facts that $\text{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and $Int[\delta, \theta]$ is the critical interval of the pentagon $\{\sigma, \gamma, \delta, \theta, \tau\}$ imply that there is a congruence Ω on **A** such that

(i) Ω is the largest congruence on **A** satisfying $\Omega|_U \subseteq B^2 \cup T^2$,
(ii) B is a $\Omega|_U$-block, and
(iii) $\theta \leq \Omega$ and $\gamma \not\leq \Omega$.

Since, $\lambda_0 = \theta \leq \Omega$, $\gamma \not\leq \Omega$, and $\gamma \leq \tau = \lambda_k$, there is an i such that $\lambda_i|_U \subseteq B^2 \cup T^2$ while $\lambda_{i+1}|_U \not\subseteq B^2 \cup T^2$.

Since λ_{i+1} is connected modulo λ_i by $\langle\lambda_i, \lambda_{i+1}\rangle$-traces (Theorem 2.8 of [5]), $\lambda_i \subseteq B^2 \cup T^2$ and $\lambda_{i+1} \not\subseteq B^2 \cup T^2$, we see that there is $\langle\lambda_i, \lambda_{i+1}\rangle$-trace M such that $M \cap B \neq$

$\emptyset \neq M \cap T$. Choose $b \in M \cap B$ and $t \in M \cap T$. We cannot have $(b, t) \notin \lambda_i$, since λ_i does not connect B to T, so $\{b, t\}$ is a $\langle \lambda_i, \lambda_{i+1} \rangle$-subtrace. From the local equation $f(x, y, y) \approx_{K_i} x$ we get $f(b, t, t) \equiv_{\lambda_i} b$, so $ef(b, t, t) \equiv_{\lambda_i} e(b) = b \in B$. (The polynomial e was fixed before the previous paragraph.) Since B is an $\Omega|_U$-block and $0, b \in B$, we get that $ef(0, t, t) \equiv_\Omega ef(b, t, t) \equiv_\Omega b$, so $ef(0, t, t) \in B$. Lemma 2.2 of [8] shows that, for any polynomial $p(x_1, \ldots, x_n) \in \mathrm{Pol}(\mathbf{A}|_U)$, if t is in the tail of U and $p(t, \ldots, t) \in B$, then $p(\theta|_U, \ldots, \theta|_U) \subseteq \delta$. Applying this fact to the binary polynomial $ef(0, x, y)$ we deduce from $ef(0, t, t) \in B$ that $ef(0, \theta|_U, \theta|_U) \subseteq \delta$. But this leads to

$$e(w) = ef(0, 0, 1) \equiv_\delta ef(0, 0, 0) \equiv_\delta e(0) = 0,$$

which contradicts our earlier conclusion that $(e(w), 0) \notin \delta$. □

LEMMA 2.3. *Let* $\mathbf{A}$ *be a finite algebra for which* $\mathrm{typ}\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. *Assume that* $\mathrm{Con}(\mathbf{A})$ *has a sublattice isomorphic to the pentagon, labeled as in Lemma* 2.2. *Let* $I = \mathit{Int}[\beta, \tau]$ *and* $J = \mathit{Int}[\alpha, \beta]$. *If* $\mathbf{A}$ *satisfies the local equations*

$$f(x, y, x) \approx_I x, \quad f(x, y, x) \approx_J x, \quad \text{and} \quad f(x, x, y) \approx_J x,$$

then $\mathbf{A}$ *satisfies the local equation* $f(x, y, y) \approx_J x$.

Proof. Assume not. Since $f(x, y, y) \not\approx_J x$, there exist a two-element subinterval $K = \mathit{Int}[\delta, \theta]$ of J, and a $\langle \delta, \theta \rangle$-subtrace $\{0, 1\}$ such that $w = f(0, 1, 1) \not\equiv_\delta 0$. We shall derive a contradiction to this.

Assume that $\mathrm{typ}(\delta, \theta) = \mathbf{4}$. Let U be a $\langle \delta, \theta \rangle$-minimal set containing $\{0, 1\}$. Since $(w, 0) \in \theta - \delta$ there is a unary polynomial k such that $k(A) = U$ and $(k(w), k(0)) \notin \delta$ (Theorem 2.8(4) of [5]). If $k(\theta|_U) \subseteq \delta$, then since $f(x, x, x) \approx_J x$ we get that

$$kf(0, 0, 0) \equiv_\delta k(0) \equiv_\delta k(1) \equiv_\delta kf(1, 1, 1).$$

The element $kf(0, 1, 1)$ is between the comparable elements $kf(0, 0, 0)$ and $kf(1, 1, 1)$ in the $\langle \delta, \theta \rangle$-preorder of the θ-block of 0. The elements $kf(0, 0, 0)$ and $kf(1, 1, 1)$ are δ-related, so $k(w) = kf(0, 1, 1) \equiv_\delta kf(0, 0, 0) \equiv_\delta k(0)$, contrary to the choice of k. Consequently $k|_U$ is a permutation. In this situation, let e be an idempotent iterate of k. The polynomial e has the properties $e(A) = U$ and $(e(w), e(0)) = (e(w), 0) \in \theta - \delta$.

If $\mathrm{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}\}$, then Lemma 4.7 of [10] guarantees the existence of an idempotent unary polynomial e with the same properties: $e(A) = U$ and $(e(w), e(0)) = (e(w), 0) \in \theta - \delta$.

The local equation $f(x, x, y) \approx_J x$ ensures that $ef(0, 0, 1) \equiv_\delta e(0) = 0$, and the choice of w together with the properties established for e ensure that $ef(0, 1, 1) = e(w) \not\equiv_\delta 0$. This shows that the polynomial $ef(0, x, 1)$ satisfies $ef(0, \theta|_U, 1) \not\subseteq \delta$, so $ef(0, x, 1)$ is a permutation of U.

As in the proof of Lemma 2.2, there exists $K_i = Int[\lambda_i, \lambda_{i+1}]$, a two-element subinterval of either I or J, for which $\lambda_i|_U \subseteq B^2 \cup T^2$ and $\lambda_{i+1}|_U \not\subseteq B^2 \cup T^2$ where B is the body and T is the tail of U. As in that proof, there is a $\langle\lambda_i, \lambda_{i+1}\rangle$-subtrace $\{b, t\}$ with $(b, t) \in (B \times T)$. The local equation $f(x, y, x) \approx_{K_i} x$ forces $ef(b, t, b) \equiv_{\lambda_i} b \in B$. Since $\lambda_i|_U \subseteq B^2 \cup T^2$ this means that $ef(b, t, b) \in B$. But $b, 0, 1 \in B$ and B is an $\Omega|_U$-block (where Ω is as defined in the proof of Lemma 2.2), so

$$ef(0, t, 1) \equiv_\Omega ef(b, t, b) \equiv_\Omega b \in B.$$

Thus $ef(0, x, 1)$ maps $t \in T$ into the body B. No polynomial permutation of U can do this, so we have contradicted the conclusion of the previous paragraph. □

THEOREM 2.4. *Let $\mathcal{V}$ be a locally finite variety. The following are equivalent.*

(1) *$\mathcal{V}$ satisfies a nontrivial congruence identity.*
(2) typ$\{\mathcal{V}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$.
(3) *There is a positive integer k and 3-ary terms $d_0, \ldots, d_{2k+1}, e_0, \ldots, e_{2k+1}, p$ such that $\mathcal{V}$ satisfies the following equations:*
   (i) $d_0(x, y, z) \approx p(x, y, z) \approx e_0(x, y, z)$;
   (ii) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$ *and* $e_i(x, x, y) \approx e_{i+1}(x, x, y)$ *for even i;*
   (iii) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$, $d_i(x, y, x) \approx d_{i+1}(x, y, x)$,
   $e_i(x, y, y) \approx e_{i+1}(x, y, y)$ *and* $e_i(x, y, x) \approx e_{i+1}(x, y, x)$ *for odd i;*
   (iv) $d_{2k+1}(x, y, z) \approx x$ *and* $e_{2k+1}(x, y, z) \approx z$.
(4) *There is a positive integer N such that no finite algebra in $\mathcal{V}$ has a herringbone of length $> N$ in its congruence lattice.*
(5) *There is a positive integer M such that $\mathcal{V}$ satisfies $\beta^M = \beta^{M+1}$ as a congruence identity. (See the Introduction for the definition of β^n.)*
(6) *There is a lattice identity ε which $\mathcal{V}$ satisfies as a congruence identity, but which fails in the lattice $\mathbf{D}_2$.*

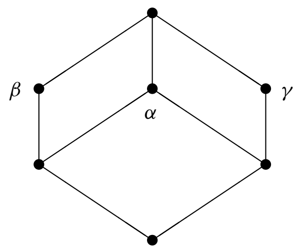
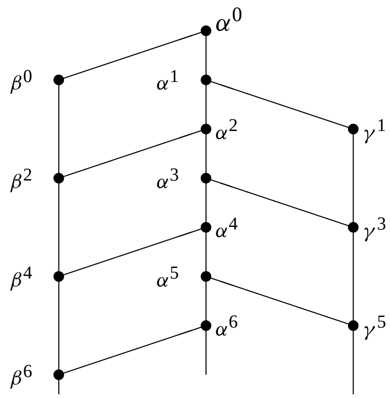


Figure 2 $\mathbf{D}_2$

Proof. The equivalence of (1) and (2) is proved in Theorem 3.7 of [7]. The equivalence of (2) and (3)′ is proved in Theorem 9.8 of [5], where (3)′ is the same as (3) with subscripts chosen differently. We will prove (2)&(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). The implication (6) $\Longrightarrow$ (1) is trivial.

We start with the proof of (2)&(3) $\Longrightarrow$ (4). We assume (2) as a hypothesis so that we are free to use Lemmas 2.2 and 2.3 as needed; otherwise we are simply arguing that (3) implies (4). Suppose that $\mathbf{A} \in \mathcal{V}$ is finite, and that Con($\mathbf{A}$) has a long herringbone labeled with α's, β's and γ's as in the following diagram.



CLAIM 2.5. *For each* $0 \leq j \leq k$ *and each* $u \geq 2j$ *the local equation*

$$d_{2k-2j}(x, y, y) \approx_K x$$

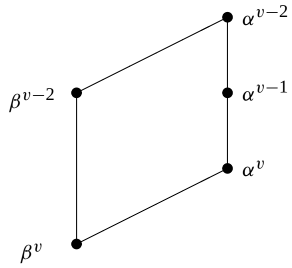
holds for $K = Int[\alpha^{u+1}, \alpha^u]$.

By (3)(ii) and (3)(iv), the algebra $\mathbf{A}$ satisfies the equations

$$d_{2k}(x, y, y) \approx d_{2k+1}(x, y, y) \quad \text{and} \quad d_{2k+1}(x, y, z) \approx x.$$

In particular, $d_{2k}(x, y, y) \approx_K x$ for each interval K in Con($\mathbf{A}$), so the claim holds when $j = 0$.

Suppose that the claim has been established for some j and all $u \geq 2j$. We argue that the claim holds for $j+1$ and all $v \geq 2(j+1) = 2j+2$. For even $v \geq 2j+2$, the congruences $\{\alpha^{v-2}, \alpha^{v-1}, \alpha^v, \beta^{v-2}, \beta^v\}$ form a pentagon in Con($\mathbf{A}$). The inductive assumption implies that $d_{2k-2j}(x, y, y) \approx_I x$ holds for $I = Int[\alpha^{v-1}, \alpha^{v-2}]$ and $d_{2k-2j}(x, y, y) \approx_J x$ holds for $J = Int[\alpha^v, \alpha^{v-1}]$, since $v, v-1 \geq 2j$.



The hypotheses of Lemma 2.2 are met, so

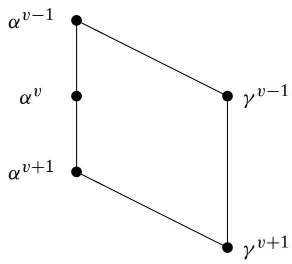
$$d_{2k-2j}(x, x, y) \approx_J x \quad \text{and} \quad d_{2k-2j}(x, y, x) \approx_J x$$

hold for $J = Int[\alpha^v, \alpha^{v-1}]$ when v is even.

By (3)(iii), **A** satisfies $d_{2k-2i-1}(x, x, y) \approx d_{2k-2i}(x, x, y)$ and $d_{2k-2i-1}(x, y, x) \approx d_{2k-2i}(x, y, x)$, so

$$d_{2k-2i-1}(x, x, y) \approx_J x \quad \text{and} \quad d_{2k-2i-1}(x, y, x) \approx_J x$$

whenever $J = Int[\alpha^v, \alpha^{v-1}]$ and $v \geq 2j + 2$ is even. This establishes the hypotheses of Lemma 2.3 for $f = d_{2k-2i-1}$ with respect to the pentagon $\{\alpha^{v-1}, \alpha^v, \alpha^{v+1}, \gamma^{v-1}, \gamma^{v+1}\}$ and the intervals $I = Int[\alpha^v, \alpha^{v-1}]$ and $J = Int[\alpha^{v+1}, \alpha^v]$.



The conclusion of that lemma is that the local equation $d_{2k-2i-1}(x, y, y) \approx_J x$ holds. By (3)(ii), **A** satisfies $d_{2k-2i-2}(x, y, y) \approx d_{2k-2i-1}(x, y, y)$, hence $d_{2k-2i-2}(x, y, y) \approx_J x$ where $J = Int[\alpha^{v+1}, \alpha^v]$ and $v \geq 2j + 2$ is even. This establishes the inductive step of the proof for even v. The proof for odd v is the same with the roles of β and γ interchanged. This proves the claim.

The claim ensures that $d_0(x, y, y) \approx_K x$ whenever $K = Int[\alpha^{u+1}, \alpha^u]$ and $u \geq 2k$. Repeating all arguments with $e_i(z, y, x)$ in place of $d_i(x, y, z)$, we also get that $e_0(x, x, y) \approx_K y$ for $K = Int[\alpha^{u+1}, \alpha^u]$ whenever $u \geq 2k$. Since **A** satisfies $d_0(x, y, z) \approx$

$p(x, y, z) \approx e_0(x, y, z)$, this means that $p(x, y, y) \approx_K x$ and $p(x, x, y) \approx_K y$ for $K = Int[\alpha^{u+1}, \alpha^u]$ whenever $u \geq 2k$.

We apply Lemma 2.2 one final time using $f(x, y, z) = p(x, y, z)$ and the pentagon $\{\alpha^u, \alpha^{u+1}, \alpha^{u+2}, \beta^u, \beta^{u+2}\}$ for some even $u \geq 2k$. (For odd u, use γ's in place of β's.) In this situation, $I = Int[\alpha^{u+1}, \alpha^u]$ and $J = Int[\alpha^{u+2}, \alpha^{u+1}]$. We have $p(x, y, y) \approx_I x$ and $p(x, y, y) \approx_J x$, so we deduce that $p(x, x, y) \approx_J x$ according to Lemma 2.2. On the other hand, we have already shown that $p(x, x, y) \approx_J y$. Thus $x \approx_J y$. Referring to the definition of $\approx_J$, we see that this means that there are no subtraces in $\alpha^{u+2} - \alpha^{u+1}$, and therefore $\alpha^{u+2} = \alpha^{u+1}$ whenever $u \geq 2k$. It follows that any herringbone must terminate at α^{2k+1} or sooner. Hence for $N = 2k + 1$, Con(**A**) has no herringbone of length $> N$.

Now we prove that (4) $\Longrightarrow$ (5). Start with three congruences α, β and γ, and (as in the Introduction) define $\beta^0 = \beta, \gamma^0 = \gamma, \beta^{n+1} = \beta \wedge (\alpha \vee \gamma^n)$, and $\gamma^{n+1} = \gamma \wedge (\alpha \vee \beta^n)$. Let $\alpha^n = \alpha \vee \beta^n$ if n is even, and $\alpha^n = \alpha \vee \gamma^n$ if n is odd. Since $\beta^0 = \beta \geq \beta \wedge (\alpha \vee \gamma^0) = \beta^1$, and $\gamma^0 \geq \gamma^1$, it is easy to see inductively that

$$\beta^{n+1} = \beta \wedge (\alpha \vee \gamma^n) \geq \beta \wedge (\alpha \vee \gamma^{n+1}) = \beta^{n+2},$$

and $\gamma^{n+1} \geq \gamma^{n+2}$. Thus the β and γ-sequences are descending chains, which forces the α-sequence to be a descending chain. We claim that

$$\{\alpha^n \mid \text{all } n\} \cup \{\beta^n \mid \text{even } n\} \cup \{\gamma^n \mid \text{odd } n\}$$

is a herringbone. To see this, we must verify conditions (β) and (γ) from the beginning of this section:

(β) $\alpha^{2i+1} \wedge \beta^{2i} = \beta^{2i+2}, \alpha^{2i+2} \vee \beta^{2i} = \alpha^{2i}$
(γ) $\alpha^{2i+2} \wedge \gamma^{2i+1} = \gamma^{2i+3}, \alpha^{2i+3} \vee \gamma^{2i+1} = \alpha^{2i+1}$.

To show that $\alpha^{2i+1} \wedge \beta^{2i} = \beta^{2i+2}$, note that

$$\beta^{2i+2} = \beta \wedge (\alpha \vee \gamma^{2i+1}) = \beta \wedge \alpha^{2i+1} \leq \alpha^{2i+1},$$

and (as observed earlier) $\beta^{2i+2} \leq \beta^{2i}$. Thus $\beta^{2i+2} \leq \alpha^{2i+1} \wedge \beta^{2i}$. Conversely,

$$\alpha^{2i+1} \wedge \beta^{2i} = (\alpha \vee \gamma^{2i+1}) \wedge \beta^{2i} \leq (\alpha \vee \gamma^{2i+1}) \wedge \beta = \beta^{2i+2}.$$

To show that $\alpha^{2i+2} \vee \beta^{2i} = \alpha^{2i}$, note that

$$\alpha^{2i+2} \vee \beta^{2i} \geq \alpha \vee \beta^{2i} = \alpha^{2i},$$

while $\alpha^{2i+2} \leq \alpha^{2i}$ and $\beta^{2i} \leq \alpha \vee \beta^{2i} = \alpha^{2i}$, so $\alpha^{2i+2} \vee \beta^{2i} \leq \alpha^{2i}$. This establishes (β), and (γ) can be established the same way.

It follows that if N bounds the length of any herringbone in any congruence lattice of a member of $\mathcal{V}$, then however we choose our original three congruences α, β and γ, the sequences defined above must satisfy $\alpha^N = \alpha^{N+1} = \alpha^{N+2} = \cdots$, and therefore $\beta^M = \beta^{M+1}$ holds for each $M > N$.

Finally we prove that (5) $\Longrightarrow$ (6). Fix $N > 2$ so that $\beta^N = \beta^{N+1}$ is a congruence identity of $\mathcal{V}$. Let $\mathcal{L}$ be the variety of lattices axiomatized by $\beta^N = \beta^{N+1}$. By the choice of N, $\mathcal{L}$ contains $\mathbf{D}_2$ and $\mathrm{Con}(\mathbf{A})$ for every $\mathbf{A} \in \mathcal{V}$. We now argue that $\mathbf{D}_2$ is a splitting lattice in $\mathcal{L}$. Recall the definition of this concept from [12]: A lattice $\mathbf{L}$ is a *splitting lattice in* $\mathcal{L}$ if there is an identity ε (the *conjugate identity*) that is satisfied by $\mathbf{K} \in \mathcal{L}$ if and only if $\mathbf{K}$ has no sublattice isomorphic to $\mathbf{L}$. (For example, the pentagon is a splitting lattice in the variety of all lattices, with conjugate identity equal to the modular law.) It is known that if $\mathbf{L}$ is a finite subdirectly irreducible lattice that is projective in $\mathcal{L}$, then $\mathbf{L}$ is a splitting lattice in $\mathcal{L}$. Clearly $\mathbf{D}_2$ is a finite subdirectly irreducible lattice. We argue now that $\mathbf{D}_2$ is projective in $\mathcal{L}$. For this we need to show that if some $\mathbf{K} \in \mathcal{L}$ has a homomorphism $\varphi : \mathbf{K} \to \mathbf{D}_2$ onto $\mathbf{D}_2$, then there is a section $\psi : \mathbf{D}_2 \to \mathbf{K}$ such that $\varphi \circ \psi(x) = x$ on $\mathbf{D}_2$.

To construct ψ from φ it will suffice to locate $\alpha^* \in \varphi^{-1}(\alpha)$, $\beta^* \in \varphi^{-1}(\beta)$, and $\gamma^* \in \varphi^{-1}(\gamma)$ such that $\{\alpha^*, \beta^*, \gamma^*\}$ generates a sublattice $\mathbf{D} \leq \mathbf{K}$ isomorphic to $\mathbf{D}_2$. (Refer to Figure 2 to see which elements of $\mathbf{D}_2$ are labeled α, β and γ.) For then φ restricts to a homomorphism from $\mathbf{D}$ onto a generating set for $\mathbf{D}_2$, hence an isomorphism from $\mathbf{D}$ onto $\mathbf{D}_2$, so we can choose $\psi = (\varphi|_{\mathbf{D}})^{-1}$.

Begin by choosing $\alpha' \in \varphi^{-1}(\alpha)$, $\beta' \in \varphi^{-1}(\beta)$, and $\gamma' \in \varphi^{-1}(\gamma)$ arbitrarily. It is possible to modify α' to $\alpha'' \in \varphi^{-1}(\alpha)$ so that α'' belongs to the interval $\mathit{Int}[\beta' \wedge \gamma', \beta' \vee \gamma']$ determined by β' and γ'. Simply take α'' to be

$$\alpha'' = (\beta' \vee \gamma') \wedge (\alpha' \vee (\beta' \wedge \gamma')).$$

Next, we would like our eventual choice for α^* to be the join of $\alpha^* \wedge \beta^*$ and $\alpha^* \wedge \gamma^*$. Therefore we replace α'' by

$$\alpha^* = (\alpha'' \wedge \beta') \vee (\alpha'' \wedge \gamma').$$

It is not hard to check that $\alpha^* \in \varphi^{-1}(\alpha)$, $\alpha^* \in \mathit{Int}[\beta' \wedge \gamma', \beta' \vee \gamma']$, and

$$\alpha^* = (\alpha^* \wedge \beta') \vee (\alpha^* \wedge \gamma').$$

Starting with α^*, $\beta' = \beta'_0$ and $\gamma' = \gamma'_0$ we begin our inductive construction of elements $(\beta')^{n+1} = \beta' \wedge (\alpha^* \vee (\gamma')^n)$ and $(\gamma')^{n+1} = \gamma' \wedge (\alpha^* \vee (\beta')^n)$. One can check inductively that $(\beta')^i \in \varphi^{-1}(\beta)$ and $(\gamma')^i \in \varphi^{-1}(\gamma)$ for all i, and

$$(\beta') = (\beta')^0 \geq (\beta')^1 \geq (\beta')^2 \geq \cdots \geq \alpha^* \wedge \beta', \tag{†}$$

and the same holds with γ in place of β. Since $(\beta')^N = (\beta')^{N+1}$ and $(\gamma')^N = (\gamma')^{N+1}$, this process terminates. We take $\beta^* = (\beta')^N$ and $\gamma^* = (\gamma')^N$. We claim that α^*, β^*, and γ^* generate a sublattice of $\mathbf{K}$ isomorphic to $\mathbf{D}_2$. To check this we must see that

(i) $\alpha^* \in Int[\beta^* \wedge \gamma^*, \beta^* \vee \gamma^*]$;
(ii) $\alpha^* \vee \beta^* = \alpha^* \vee \gamma^* = \beta^* \vee \gamma^*$;
(iii) $\alpha^* = (\alpha^* \wedge \beta^*) \vee (\alpha^* \wedge \gamma^*)$; and
(iv) $(\alpha^* \wedge \beta^*) \vee \gamma^* = \alpha^* \vee \beta^* \vee \gamma^* = (\alpha^* \wedge \gamma^*) \vee \beta^*$.

It is a consequence of (†) that $\beta' \geq \beta^* \geq \alpha^* \wedge \beta'$. Therefore

$$\alpha^* \wedge \beta' \geq \alpha^* \wedge \beta^* \geq \alpha^* \wedge (\alpha^* \wedge \beta') = \alpha^* \wedge \beta',$$

so $\alpha^* \wedge \beta^* = \alpha^* \wedge \beta'$. Similarly $\alpha^* \wedge \gamma^* = \alpha^* \wedge \gamma'$. This shows that $(\alpha^* \wedge \beta^*) \vee (\alpha^* \wedge \gamma^*) = \alpha^*$, so (iii) holds. We have $\alpha^* \vee \beta^* = \alpha^* \vee \beta^N \geq \gamma^{N+1} = \gamma^N = \gamma^*$, and by the same argument $\alpha^* \vee \gamma^* \geq \beta^*$. Moreover,

$$\beta^* \vee \gamma^* \geq (\alpha^* \wedge \beta^*) \vee (\alpha \wedge \gamma^*) = \alpha^*.$$

This shows that (ii) holds. Since $\alpha^* \leq \beta^* \vee \gamma^*$ and $\beta^* \wedge \gamma^* \leq \beta' \wedge \gamma' \leq \alpha^*$, we get that (i) holds. To verify (iv), note that

$$(\alpha^* \wedge \beta^*) \vee \gamma^* \geq (\alpha^* \wedge \beta^*) \vee (\alpha^* \wedge \gamma^*) = \alpha^*,$$

so $(\alpha^* \wedge \beta^*) \vee \gamma^* \geq \alpha^* \vee \gamma^* = \alpha^* \vee \beta^* \vee \gamma^*$. Similarly $(\alpha^* \wedge \gamma^*) \vee \beta^* \geq \alpha^* \vee \beta^* \vee \gamma^*$. Thus, $\{\alpha^*, \beta^*, \gamma^*\}$ generates a sublattice of $\mathbf{K}$ that φ maps isomorphically onto $\mathbf{D}_2$. This proves that $\mathbf{D}_2$ is projective in $\mathcal{L}$. (Remark: $\mathbf{D}_2$ is not projective in the class of all lattices, hence not a splitting lattice in this class, since it is not semidistributive. See [12].)

Let ε be the splitting equation for $\mathbf{D}_2$ in $\mathcal{L}$. By Theorem 9.8 of [5], $\mathbf{D}_2$ is not isomorphic to a sublattice of Con($\mathbf{A}$) for any finite $\mathbf{A} \in \mathcal{V}$. Since each lattice Con($\mathbf{A}$), $\mathbf{A}$ finite, is a member of $\mathcal{L}$ by part (5) of this theorem, it follows that ε holds in the congruence lattices of finite members of $\mathcal{V}$. But the satisfaction of any particular congruence identity is a local property (see [14] or [15]), which means that if the finitely generated algebras satisfy ε as a congruence identity, then all algebras in $\mathcal{V}$ satisfy ε as a congruence identity. Thus, ε is a lattice identity that fails in $\mathbf{D}_2$ but is a congruence identity of $\mathcal{V}$. □

The result stated in the title of the paper is part of the next theorem.

THEOREM 2.6. *Let $\mathcal{V}$ be a locally finite variety. The following conditions are equivalent.*

(1) $typ\{\mathcal{V}\} \subseteq \{\mathbf{3}, \mathbf{4}\}$.
(2) *for some N, $\mathcal{V}$ satisfies*

$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N), \qquad E^N:$$

as a congruence identity.

(3) $\mathcal{V}$ *is congruence join semidistributive.*
(4) $\mathcal{V}$ *is congruence meet semidistributive and satisfies a nontrivial congruence identity.*

Proof. As observed in the Introduction, the satisfaction of E^n implies join semidistributivity in any lattice, so (2) $\Longrightarrow$ (3). Exercise 6.23.12 of [5] sketches the proof that congruence join semidistributivity implies congruence meet semidistributivity in any variety. Thus (3) implies the first statement in (4). Theorem 9.11 of [5] proves that the finite algebras in $\mathcal{V}$ have join semidistributive congruence lattices if and only if $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{3}, \mathbf{4}\}$. This and Theorem 2.4 (2) $\Longrightarrow$ (1) imply that condition (3) of this theorem implies the second statement in condition (4). Thus (3) $\Longrightarrow$ (4). The implication (4) $\Longrightarrow$ (1) follows from Theorems 9.10 and 9.18 of [5]. What remains to show is that (1) $\Longrightarrow$ (2).

Since $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{3}, \mathbf{4}\}$ if and only if the finite algebras in $\mathcal{V}$ have join semidistributive congruence lattices, it follows from the result of Jónsson and Rival mentioned in the Introduction that for each finite $\mathbf{A} \in \mathcal{V}$ there is some n such that $\mathrm{Con}(\mathbf{A})$ satisfies E^n:

$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta^n) \wedge (\alpha \vee \gamma^n).$$

However, the E^n's get weaker as n increases, and since $\mathcal{V}$ satisfies the congruence identity $\beta^N = \beta^{N+1}$ $(= \beta^{N+2} = \cdots)$ for some N, it follows that there is a fixed N such that all finite algebras in $\mathcal{V}$ satisfy E^N,

$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N),$$

as a congruence identity. But the satisfaction of any particular congruence identity is a local property, so all algebras in $\mathcal{V}$ satisfy E^N as a congruence identity. □

The equivalence of (1) and (3) of this theorem justifies our affirmative answer to the question of Hobby and McKenzie in the Introduction. This equivalence also explains our answer to Czédli's question, since it is shown in Theorem 9.11 of [5] that condition (1) of this theorem can be characterized by a three-variable Mal'tsev condition.

For the next corollary, if $\mathcal{K}$ is a class of algebras then $\mathrm{Con}(\mathcal{K}) = \{\mathrm{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{K}\}$ is the class of congruence lattices of members of $\mathcal{K}$, $\mathrm{Sub}(\mathrm{Con}(\mathcal{K}))$ is the class of sublattices of $\mathrm{Con}(\mathcal{K})$, and $\mathrm{CON}(\mathcal{K})$ is the variety generated by $\mathrm{Con}(\mathcal{K})$ (i.e., the *congruence variety* of $\mathcal{K}$). We now prove that join semidistributive congruence varieties are determined by the splitting lattices they contain.

COROLLARY 2.7. *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be locally finite congruence join semidistributive varieties. The following conditions are equivalent.*

(1) $\mathrm{CON}(\mathcal{V}) \subseteq \mathrm{CON}(\mathcal{W})$.
(2) *Any splitting lattice in* $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$ *is in* $\mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$.
(3) $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin})) \subseteq \mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$.

Proof. ($\neg(2) \Longrightarrow \neg(1)$): Assume that some splitting lattice $\mathbf{L}$ is embeddable in some member of $\mathrm{Con}(\mathcal{V}_{fin})$ but in no member of $\mathrm{Con}(\mathcal{W}_{fin})$. Then the congruence lattices of members of $\mathcal{W}_{fin}$ satisfy the conjugate identity for $\mathbf{L}$, and so $\mathrm{CON}(\mathcal{W})$ satisfies this identity. But $\mathrm{CON}(\mathcal{V})$ does not satisfy the conjugate identity since $\mathbf{L} \in \mathrm{CON}(\mathcal{V})$ and $\mathbf{L}$ does not satisfy its own conjugate identity. Thus $\mathrm{CON}(\mathcal{V}) \not\subseteq \mathrm{CON}(\mathcal{W})$.

($\neg(3) \Longrightarrow \neg(2)$): Assume that (3) fails. Let $\mathcal{K}$ be the class of subdirectly irreducible lattices that are homomorphic images of lattices in $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$, and let $\mathrm{P}_s(\mathcal{K})$ be the class of subdirect products of members of $\mathcal{K}$. If $\mathcal{K} \subseteq \mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$, then since $\mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$ is closed under the formation of sublattices and finite products it follows from the subdirect representation theorem that

$$\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin})) \subseteq \mathrm{P}_s(\mathcal{K}) \subseteq \mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin})),$$

contrary to the assumption that (3) fails. Thus $\mathcal{K} \not\subseteq \mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$, and so there is a subdirectly irreducible lattice $\mathbf{L} \in \mathcal{K} - \mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$ that is a homomorphic image of some lattice in $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$. Since $\mathcal{V}$ is congruence join semidistributive, the lattices in $\mathrm{Con}(\mathcal{V}_{fin})$ are finite bounded homomorphic images of free lattices (according to Corollary 27 of [3]). The class of finite bounded homomorphic images of free lattices is closed under the formation of homomorphic images, so $\mathbf{L}$ is a finite bounded subdirectly irreducible lattice. By one of the main results of [12], $\mathbf{L}$ is a splitting lattice. Splitting lattices are projective in the class of all lattices, so $\mathbf{L}$ is a projective homomorphic image of some lattice in $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$. It follows that $\mathbf{L}$ is actually a member of $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$. This shows that if (3) fails, then there is a splitting lattice $\mathbf{L}$ that belongs to $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$ but not to $\mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$.

($(3) \Longrightarrow (1)$): If the class $\mathrm{Sub}(\mathrm{Con}(\mathcal{V}_{fin}))$ is contained in the class $\mathrm{Sub}(\mathrm{Con}(\mathcal{W}_{fin}))$, then the varieties they generate are related in the same way. Since the satisfaction of any given congruence identity is a local property, the generated varieties are $\mathrm{CON}(\mathcal{V})$ and $\mathrm{CON}(\mathcal{W})$ respectively. □

ADDITIONAL REMARKS. We showed in Theorem 2.6 that whenever $\mathcal{V}$ is congruence join semidistributive, then the variety $\mathrm{CON}(\mathcal{V})$ is also join semidistributive. The analogous remark for meet semidistributivity is not true. The simplest counterexample is $\mathcal{V} =$ the variety of semilattices, which is congruence meet semidistributive but $\mathrm{CON}(\mathcal{V})$ is the variety of all lattices (according to [4]). A more surprising example appears in [13]: it is shown that if $\mathcal{V}$ is the Polin product of two copies of the variety of distributive lattices, then $\mathcal{V}$ is congruence join and meet semidistributive, $\mathrm{CON}(\mathcal{V})$ is join semidistributive (in fact, satisfies E^2), but $\mathrm{CON}(\mathcal{V})$ is not meet semidistributive. In particular, this $\mathcal{V}$ is an example of a variety that is congruence join semidistributive but has no finite bound on the length of dual herringbones.

We can produce new congruence identities from the congruence identity $\beta^M = \beta^{M+1}$ of Theorem 2.4 by using ideas from the proof of Theorem 2.6. Let $\mathcal{V}$ be a locally finite variety with typ$\{\mathcal{V}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. If $\mathbf{A} \in \mathcal{V}$ is finite, then by Theorem 7.7 (3) of [5] the quotient $\mathrm{Con}(\mathbf{A})/\overset{s}{\sim}$ of the congruence lattice by the solvability congruence is join semidistributive. Satisfaction of the equation $\beta^M = \beta^{M+1}$ is inherited by $\mathrm{Con}(\mathbf{A})/\overset{s}{\sim}$ from $\mathrm{Con}(\mathbf{A})$, so (as in the proof of Theorem 2.6) there is some N such that $\mathrm{Con}(\mathbf{A})/\overset{s}{\sim}$ satisfies E^N for all finite $\mathbf{A} \in \mathcal{V}$. Thus, any interval in $\mathrm{Con}(\mathbf{A})$ defined by an E^N-failure, $\mathit{Int}[\alpha \vee (\beta \wedge \gamma), (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N)]$, is a solvable interval. Since $\mathbf{1} \notin$ typ$\{\mathcal{V}\}$, solvable intervals consist of permuting congruences. So let $\omega_1(x_1, \ldots, x_k) \approx \omega_2(x_1, \ldots, x_k)$ be any lattice identity that holds in every lattice of permuting equivalence relations. We produce from this a $(k+3)$-ary congruence identity of $\mathcal{V}$ expressing the fact that any interval defined by an E^N-failure satisfies $\omega_1 \approx \omega_2$. The identity is constructed as follows. Start with lattice variables $\alpha, \beta, \gamma, \delta_1, \ldots, \delta_k$. Define β^i and γ^i as described earlier in the paper. Let $\mu = \alpha \vee (\beta \wedge \gamma)$ and $\nu = (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N)$. Then $I = \mathit{Int}[\mu, \nu]$ represents a typical interval defined by an E^N-failure. Let $\delta_i^* = \mu \vee (\delta_i \wedge \nu)$. The polynomial $e(x) = \mu \vee (x \wedge \nu)$ of the free lattice generated by $\{\alpha, \beta, \gamma, \delta_1, \ldots, \delta_k\}$ is idempotent and has range I. This means that the words $\delta_1^*, \ldots, \delta_k^*$ represent k typical elements of I. Thus

$$\omega_1(\delta_1^*, \ldots, \delta_k^*) \approx \omega_2(\delta_1^*, \ldots, \delta_k^*)$$

is a lattice equation that holds in a lattice if and only if $\omega_1 \approx \omega_2$ holds in all intervals defined by E^N-failures. For large enough N, this identity holds in $\mathrm{Con}(\mathbf{A})$ for all finite $\mathbf{A} \in \mathcal{V}$, hence it holds throughout $\mathcal{V}$. This shows that if $\mathcal{V}$ omits types **1** and **5**, then there is an N such that intervals in congruence lattices defined by E^N-failures satisfy all identities true in every lattice of permuting equivalence relations.

But rather than observe that intervals defined by E^N-failures *are shaped* like intervals of permuting equivalence relations, it is better to observe that they *are* intervals of permuting equivalence relations. In fact, all such intervals are locally solvable (and therefore consist of permuting congruences, by Theorem 7.12 of [5]). The reason that this is true is that if $\mathbf{A} \in \mathcal{V}$ is infinite and has congruences α, β and γ, $\mu = \alpha \vee (\beta \wedge \gamma)$ and $\nu = (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N)$, and if $\nu - \mu$ contains a 2-snag, then it is not hard to show that for some finitely generated subalgebra $\mathbf{B} \leq \mathbf{A}$ it is the case that

$$(\alpha|_B \vee (\beta|_B)^N) \wedge (\alpha|_B \vee (\gamma|_B)^N) - \alpha|_B \vee (\beta|_B \wedge \gamma|_B)$$

contains a 2-snag (the same one). Thus, the value of N that ensures that intervals of the form $\mathit{Int}[\alpha \vee (\beta \wedge \gamma), (\alpha \vee \beta^N) \wedge (\alpha \vee \gamma^N)]$ are solvable for finite algebras in $\mathcal{V}$ is a value that ensures that such intervals are locally solvable for infinite algebras in $\mathcal{V}$. In particular, this shows that intervals in $\mathrm{Con}(\mathbf{A})$ defined by failures of the join semidistributive law are locally solvable when $\mathbf{A}$ generates a locally finite variety that omits types **1** and **5**.

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