Axiomatizable and Nonaxiomatizable Congruence Prevarieties

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ABSTRACT. If $\mathcal{V}$ is a variety of algebras, let $\mathcal{L}(\mathcal{V})$ denote the prevariety of all lattices embeddable in congruence lattices of algebras in $\mathcal{V}$. We give some criteria for the first-order axiomatizability or nonaxiomatizability of $\mathcal{L}(\mathcal{V})$. One corollary to our results is a nonconstructive proof that every congruence n-permutable variety satisfies a nontrivial congruence identity.

1. Introduction

For a variety $\mathcal{V}$, let $\mathcal{L}(\mathcal{V})$ denote the class of lattices embeddable in congruence lattices of algebras in $\mathcal{V}$. It is evident that $\mathcal{L}(\mathcal{V})$ is closed under the formation of isomorphic lattices and sublattices. It is also closed under the formation of products, because a product of congruence lattices of algebras $\mathbf{A}_i \in \mathcal{V}$ is embeddable in the congruence lattice of the product $\prod_{i\in I} \mathbf{A}_i$ via the map

$$\prod_{i\in I} \mathbf{Con}(\mathbf{A}_i) \to \mathbf{Con}\Big(\prod_{i\in I} \mathbf{A}_i\Big) : (\gamma_i)_{i\in I} \mapsto \Gamma,$$

where $\mathbf{a}\,\Gamma\,\mathbf{b}$ if $a_i\ \gamma_i\ b_i$ for all i and $a_j = b_j$ for all but finitely many j. Thus $\mathcal{L}(\mathcal{V})$ a prevariety, which we call the **congruence prevariety** of $\mathcal{V}$. In this note we discuss the first-order axiomatizability of $\mathcal{L}(\mathcal{V})$. Theorem 2.1 describes some conditions sufficient to guarantee the first-order axiomatizability of $\mathcal{L}(\mathcal{V})$, and Theorems 3.1 and 3.2 indicate some conditions necessary for axiomatizability. The combination of Theorems 2.1 and 3.1 yields an unexpected new proof that every congruence n-permutable variety satisfies a nontrivial congruence identity.

We use the symbol $+$ for lattice join and juxtaposition or $\cdot$ for lattice meet. Meet takes precedence over join in expressions that are not fully parenthesized.

2. Axiomatizable Congruence Prevarieties

Our only result concerning axiomatizable congruence prevarieties is a summary of what can be derived from the literature.

Theorem 2.1. *If $\mathcal{V}$ satisfies any one of the following conditions, then $\mathcal{L}(\mathcal{V})$ is axiomatizable.*

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(1) *$\mathcal{V}$ is congruence distributive.*
(2) *$\mathcal{V}$ is congruence n-permutable for some n.*
(3) *$\mathcal{V}$ contains a nontrivial finite strongly solvable algebra.*

Proof. For item (1), any prevariety of distributive lattices is a variety, hence is first-order axiomatizable.

For item (3), if $\mathcal{V}$ contains a nontrivial finite strongly solvable algebra, then $\mathcal{V}$ contains a locally finite, locally solvable, minimal subvariety $\mathcal{M}$. According to the results of [8] or [15], $\mathcal{M}$ is term equivalent to the variety of sets or pointed sets. In either case, the congruence lattices of members of $\mathcal{M}$ are exactly the partition lattices. Since every lattice is embeddable in a partition lattice, $\mathcal{L}(\mathcal{M})$ (and therefore also $\mathcal{L}(\mathcal{V})$) is the variety of all lattices. Since the prevariety $\mathcal{L}(\mathcal{V})$ is a variety, it is first-order axiomatizable.

Item (2) is proved in both [1] and [5]. We include a proof here, too. To show that the prevariety $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable, it suffices to show that it is closed under ultraproducts. That this is so is a consequence of the following claim.

Claim 2.2. *Let $\mathbf{A}_i$, $i \in I$, be similar algebras with n-permuting congruences. If $\mathcal{U}$ is an ultrafilter on I, then the ultraproduct $\prod_U \mathbf{Con}(\mathbf{A}_i)$ is embeddable in $\mathbf{Con}\left(\prod_U \mathbf{A}_i\right)$.*

Let L be the common language of the $\mathbf{A}_i$'s. Expand L to a language L^+ containing extra predicate symbols, as follows. For each sequence $\Theta := (\theta_i)_{i\in I} \in \prod_I \mathbf{Con}(\mathbf{A}_i)$ introduce a binary predicate symbol $\Theta(x, y)$. Interpret $\Theta(x, y)$ in $\mathbf{A}_i$ so that $\Theta^{\mathbf{A}_i}(a, b)$ is true iff $(a, b) \in \theta_i$. Each $\mathbf{A}_i$ is an L^+-structure, so the ultraproduct $\mathbf{A} := \prod_U \mathbf{A}_i$ is also an L^+-structure. The fact that $\Theta^{\mathbf{A}_i}(x, y)$ defines a congruence on $\mathbf{A}_i$ is first-order expressible, so $\Theta^{\mathbf{A}}(x, y)$ defines a congruence on $\mathbf{A}$. Consider the assignment $\prod_U \mathbf{Con}(\mathbf{A}_i) \to \mathbf{Con}(\mathbf{A})$ defined by

$$(\theta_i)_{i\in I}/\mathcal{U} \; (= \; \Theta/\mathcal{U}) \mapsto \text{the congruence defined by } \Theta^{\mathbf{A}}(x, y). \tag{2.1}$$

This is a well defined mapping, since if $\Theta = (\theta_i)_{i\in I}$ equals $\Psi = (\psi_i)_{i\in I}$ almost everywhere modulo $\mathcal{U}$, then $\mathbf{A}_i$ satisfies the sentence $\forall x, y(\Theta(x, y) \leftrightarrow \Psi(x, y))$ for almost all i, so $\mathbf{A}$ also satisfies this sentence. In this situation $\Theta^{\mathbf{A}}(x, y)$ and $\Psi^{\mathbf{A}}(x, y)$ define the same relation on $\mathbf{A}$. If $(\Theta \cdot \Psi)(x, y)$ is the predicate associated to the lattice meet $(\theta_i)_{i\in I} \cdot (\psi)_{i\in I} = (\theta_i \cdot \psi_i)_{i\in I}$, then

$$\mathbf{A}_i \models (\Theta \cdot \Psi)(x, y) \leftrightarrow \Theta(x, y) \;\&\; \Psi(x, y)$$

for every $i \in I$. Therefore $(\Theta \cdot \Psi)^{\mathbf{A}}(a, b)$ holds iff $\Theta^{\mathbf{A}}(a, b)$ and $\Psi^{\mathbf{A}}(a, b)$ both hold, proving that the assignment (2.1) preserves the lattice meet. If all $\mathbf{A}_i$ have n-permuting congruences, then $(\theta_i)_{i\in I} + (\psi_i)_{i\in I} = (\theta_i + \psi_i)_{i\in I} =: \Theta + \Psi$, and

$$\begin{aligned} \mathbf{A}_i \models \quad & (\Theta + \Psi)(x, y) \leftrightarrow \exists z_0, \dots, z_n(x = z_0 \;\&\; y = z_n \;\& \\ & \Theta(z_i, z_{i+1}),\ i \text{ even, and } \ \Psi(z_i, z_{i+1}),\ i \text{ odd}) \end{aligned} \tag{2.2}$$

for all i. Thus $\mathbf{A}$ satisfies the formula in (2.2). It follows that the congruence defined by $(\Theta + \Psi)^{\mathbf{A}}(x, y)$ is the n-fold composition (hence the join) of the congruences defined by $\Theta^{\mathbf{A}}(x, y)$ and $\Psi^{\mathbf{A}}(x, y)$. This fact implies that (2.1) preserves the lattice join, completing the proof of the claim and the theorem. ∎

Given that congruence modularity is congruence distributivity composed with congruence permutability, [3], it is reasonable to ask

Problem 2.3. Is the congruence prevariety of a congruence modular variety first-order axiomatizable?

3. Nonaxiomatizable Congruence Prevarieties

This section contains the two main results of the paper, which are theorems about the nonaxiomatizability of $\mathcal{L}(\mathcal{V})$ under some circumstances. The theorems will be stated in the form "If $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{V}$ satisfies a certain Maltsev condition, then $\mathcal{V}$ must satisfy a stronger Maltsev condition". Statements of this form, $A \ \& \ M \implies M^+$, which are phrased in terms of axiomatizability, may be written in terms of nonaxiomatizability as $M \ \& \ \neg M^+ \implies \neg A$.

To express our theorems we need some notation for lattice words in the variables x, y and z. Let $\beta_0(x,y,z) = y$, $\gamma_0(x,y,z) = z$, $\beta_{k+1}(x,y,z) = y + x \cdot \gamma_k(x,y,z)$, and $\gamma_{k+1}(x,y,z) = z + x \cdot \beta_k(x,y,z)$. Let $\overline{x} = x(y+z) + yz$ and $\overline{\overline{x}} = (x+y)(x+z)(y+z)$. Throughout this section we shall focus on three related sequences of lattice identities, namely

$$\beta_m(x,y,z) \approx \beta_{m+1}(x,y,z), \qquad (\varepsilon_m)$$

$$\beta_m(\overline{x},y,z) \approx \beta_{m+1}(\overline{x},y,z), \quad \text{and} \qquad (\overline{\varepsilon}_m)$$

$$\beta_m(\overline{\overline{x}},y,z) \approx \beta_{m+1}(\overline{\overline{x}},y,z). \qquad (\overline{\overline{\varepsilon}}_m)$$

For a fixed $m > 0$, these identities get weaker as more bars are added to ε, and for a fixed number of bars the identities get weaker as m increases. (These assertions will be clearer after the discussion preceding the proof of Theorem 3.2.)

The two main theorems of this paper are the following.

Theorem 3.1. *If $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{V}$ satisfies any nontrivial idempotent Maltsev condition, then $\mathcal{L}(\mathcal{V})$ satisfies identity $\overline{\overline{\varepsilon}}_m$ for some m.*

Theorem 3.2. *If $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and satisfies identity $\overline{\varepsilon}_m$ for some m, then $\mathcal{L}(\mathcal{V})$ also satisfies identity ε_M for some M.*

It will become clearer later that $\overline{\overline{\varepsilon}}_m$ is a nontrivial lattice identity for all m, so Theorem 3.1 has content. As for Theorem 3.2, it is not difficult to construct *lattices* satisfying (say) identity $\overline{\varepsilon}_1$ and not satisfying ε_M for any M, but we know of no variety $\mathcal{V}$ such that $\mathcal{L}(\mathcal{V})$ satisfies $\overline{\varepsilon}_m$ for some m and does not satisfy ε_M for any M. Hence we pose a problem.

Problem 3.3. Does Theorem 3.2 have content? (Is there a variety $\mathcal{V}$ such that $\mathcal{L}(\mathcal{V})$ satisfies $\overline{\varepsilon}_m$ for some m but does not satisfy ε_M for any M?)

Before proving Theorems 3.1 and 3.2 we derive some corollaries.

Corollary 3.4. *If $\mathcal{V}$ is congruence n-permutable, then $\mathcal{V}$ satisfies a nontrivial congruence identity.*

Proof. Any congruence n-permutable variety $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition, e.g. the one in [4]. By Theorem 2.1 (2), $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable, so it follows from Theorem 3.1 that $\mathcal{L}(\mathcal{V}) \models \overline{\overline{\varepsilon}}_m$ for some m. □

In fact, Paolo Lipparini has recently shown in [14] that if $\mathcal{V}$ is congruence n-permutable for some n, then $\mathcal{L}(\mathcal{V})$ satisfies the identity ε_m for some m, which is a stronger identity than the identity $\overline{\overline{\varepsilon}}_m$ guaranteed in Corollary 3.4. We do not know whether $\mathcal{L}(\mathcal{V})$ must satisfy ε_m for some m assuming only that $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition, so we pose this as a problem.

Problem 3.5. Is it true that $\mathcal{L}(\mathcal{V}) \models \varepsilon_m$ for some m whenever $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition?

Problem 3.5 could be answered affirmatively by answering Problem 3.3 negatively. Or, given Theorems 3.1 and 3.2, Problem 3.5 could be answered affirmatively by showing that if $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{L}(\mathcal{V}) \models \overline{\overline{\varepsilon}}_m$ for some m, then $\mathcal{L}(\mathcal{V}) \models \overline{\varepsilon}_M$ for some M.

Corollary 3.6. *If $\mathcal{V}$ is a locally finite variety for which $\mathbf{1} \notin \operatorname{typ}\{\mathcal{V}\}$ and $\mathbf{5} \in \operatorname{typ}\{\mathcal{V}\}$, then $\mathcal{L}(\mathcal{V})$ is not first-order axiomatizable.*

Proof. By Theorem 9.6 of [6], $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition iff $\mathbf{1} \notin \operatorname{typ}\{\mathcal{V}\}$. By Theorem 9.18 of [6], if $\mathcal{V}$ satisfies a nontrivial congruence identity, then $\mathbf{5} \notin \operatorname{typ}\{\mathcal{V}\}$. Thus, if $\mathbf{1} \notin \operatorname{typ}\{\mathcal{V}\}$ and $\mathbf{5} \in \operatorname{typ}\{\mathcal{V}\}$, then $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition but no nontrivial congruence identity. Theorem 3.1 proves that $\mathcal{L}(\mathcal{V})$ is not first-order axiomatizable. □

For example, the variety $\mathcal{S}$ of semilattices is locally finite and satisfies $\operatorname{typ}\{\mathcal{S}\} = \{\mathbf{5}\}$, so $\mathcal{L}(\mathcal{S})$ is not first-order axiomatizable.

Next we embark on the proof Theorem 3.1. The first lemma allows us to view one of the hypotheses of Theorem 3.1 in a lattice-theoretic way.

Lemma 3.7. *Let $\mathcal{V}$ be a variety. The following conditions are equivalent.*

(1) *$\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition.*
(2) *$\mathbf{D}_1$ is not in $\mathcal{L}(\mathcal{V})$.*

Proof. This is proved in Theorem 4.23 of [7]. □

Thus, Theorem 3.1 may be viewed as asserting that if $L(\mathcal{V})$ is first-order axiomatizable, then either $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$ or $\mathcal{L}(\mathcal{V}) \models \overline{\overline{\varepsilon}}_m$ for some m. In order to recognize if $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$, it will be useful to have a presentation of $\mathbf{D}_1$.

Lemma 3.8. *A presentation of $\mathbf{D}_1$ relative to the variety of all lattices is $\langle G \mid R \rangle$ where $G = \{x, y, z\}$ and R consists of the relations:*

(I) $x \leq y + z$,
(II) $z(x + y) \leq y$,
(III) $y(x + z) \leq z$, *and*
(IV) $(x + y)(x + z) \leq x$.

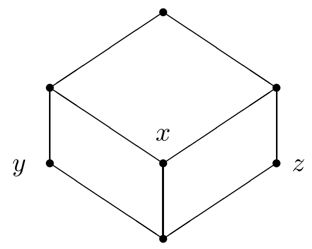

Figure 1. The lattice $\mathbf{D}_1$

Moreover any lattice generated by G and satisfying the relations in R and also satisfying $x \not\leq z$ is isomorphic to $\mathbf{D}_1$.

Proof. This can be derived from Lemma 5.27 of [7], which is the dual of the Lemma 3.8. □

Proof of Theorem 3.1. We shall show that if $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and fails to satisfy $\overline{\overline{\varepsilon}}_m$ for all m, then $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$. The result then follows from Lemma 3.7.

Let $\mathbf{F}$ be the lattice that is free relative to $\mathcal{L}(\mathcal{V})$ over the set $\{a, b, c\}$. Choose a nonprincipal ultrafilter $\mathcal{U}$ on ω, and let $\mathbf{F}^*$ denote the ultrapower $\prod_{\mathcal{U}} \mathbf{F}$. Let $\Delta\colon \mathbf{F} \to \mathbf{F}^*\colon x \mapsto (x, x, \ldots)/\mathcal{U}$ be the diagonal embedding of $\mathbf{F}$ into $\mathbf{F}^*$. Since $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable, $\mathbf{F}^* \in \mathcal{L}(\mathcal{V})$, hence we may (and do) consider $\mathbf{F}^*$ to be a sublattice of $\mathbf{Con}(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{V}$.

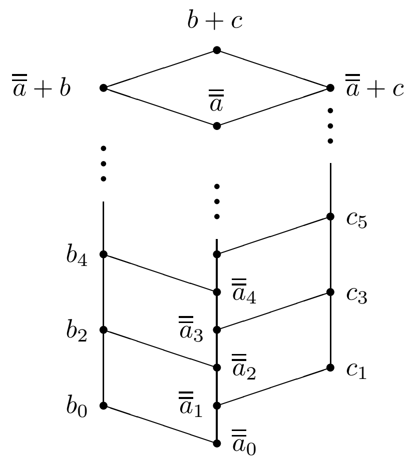

Figure 2. Some elements of $\mathbf{F}$

Let $\overline{\overline{a}} = (a + b)(a + c)(b + c)$, $b_k = \beta_k(\overline{\overline{a}}, b, c)$, $c_k = \gamma_k(\overline{\overline{a}}, b, c)$, $\overline{\overline{a}}_k = \overline{\overline{a}} \cdot b_k$ if k is even and $\overline{\overline{a}}_k = \overline{\overline{a}} \cdot c_k$ if k is odd. The elements $\overline{\overline{a}}, \overline{\overline{a}}_k, b_k$ and c_k all belong to $\mathbf{F}$, and some of them are ordered as depicted in Figure 2. From the way the elements $\overline{\overline{a}}, b_k$, and c_k are defined in terms of a, b and c, and the fact that $\overline{\overline{a}} \leq b + c$, it is easy to see that the following relations hold in $\mathbf{F}$.

(i) $b = b_0 \leq b_1 \leq b_2 \leq b_3 \leq \cdots \leq \overline{\overline{a}} + b \leq b + c$,
(ii) $c = c_0 \leq c_1 \leq c_2 \leq c_3 \leq \cdots \leq \overline{\overline{a}} + c \leq b + c$,
(iii) $\overline{\overline{a}}_0 \leq \overline{\overline{a}}_1 \leq \overline{\overline{a}}_2 \leq \cdots \leq \overline{\overline{a}}$,
(iv) $b + \overline{\overline{a}}_{2k+1} = b + \overline{\overline{a}}_{2k+2} = b_{2k+2}$,
(v) $c + \overline{\overline{a}}_{2k} = c + \overline{\overline{a}}_{2k+1} = c_{2k+1}$,
(vi) $b_{2k} \cdot \overline{\overline{a}} = \overline{\overline{a}}_{2k}$, and
(vii) $c_{2k+1} \cdot \overline{\overline{a}} = \overline{\overline{a}}_{2k+1}$.

Less obvious is the fact that

(viii) $\overline{\overline{a}} = (\overline{\overline{a}} + b)(\overline{\overline{a}} + c)$.

To see that this is so, observe that $\overline{\overline{a}} \leq \overline{\overline{a}} + b \leq (a + b)(b + c)$ and $\overline{\overline{a}} \leq \overline{\overline{a}} + c \leq (a + c)(b + c)$, so meeting corresponding elements in these inequalities yields

$$\overline{\overline{a}} = \overline{\overline{a}} \cdot \overline{\overline{a}} \leq (\overline{\overline{a}} + b)(\overline{\overline{a}} + c) \leq (a + b)(a + c)(b + c) = \overline{\overline{a}}.$$

This shows that (viii) holds.

Using these relations it can be seen that the order among the elements is the one that is depicted in Figure 2, and also that if any two of the elements that appear in the figure are equal in $\mathbf{F}$, then $b_k = b_{k+1}$ for all sufficiently large k. If this happens, then since $\mathbf{F}$ is freely generated by $\{a, b, c\}$ we get that $\mathcal{L}(\mathcal{V})$ satisfies $\beta_k(\overline{\overline{x}}, y, z) \approx \beta_{k+1}(\overline{\overline{x}}, y, z)$ for any sufficiently large k. Therefore, if $\mathcal{L}(\mathcal{V})$ fails to satisfy $\beta_k(\overline{\overline{x}}, y, z) \approx \beta_{k+1}(\overline{\overline{x}}, y, z)$ for every k, then all elements in Figure 2 are distinct. To finish the proof, we must derive from this property that $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$.

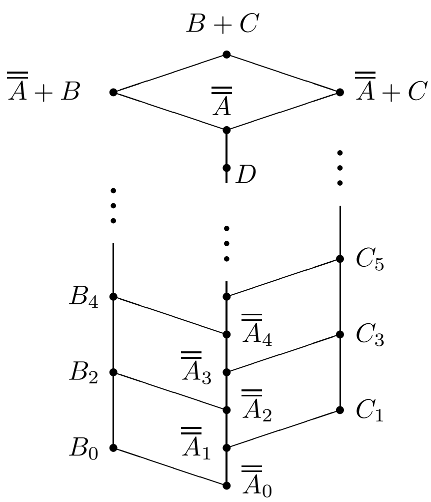

FIGURE 3. Some elements of $\mathbf{F}^*$

Assume that all elements of Figure 2 are distinct. For each $w \in F$ that is denoted by a lower case letter, use the corresponding upper case letter W to denote the element of $\mathbf{F}^*$ that is the image of w under the diagonal embedding. That is, $W := \Delta(w) = (w, w, w, \ldots)/\mathcal{U}$. Corresponding to the elements $a, \overline{a}, \overline{\overline{a}}_k, b, b_k, c, c_k \in F$ we therefore have $A, \overline{\overline{A}}, \overline{\overline{A}}_k, B, B_k, C, C_k \in F^*$. Define $D := \left(\overline{\overline{a}}_0, \overline{\overline{a}}_1, \overline{\overline{a}}_2, \cdots\right)/\mathcal{U} \in F^*$. The fact that each coordinate of $\left(\overline{\overline{a}}_0, \overline{\overline{a}}_1, \overline{\overline{a}}_2, \cdots\right)$ is strictly less than the corresponding coordinate of $\left(\overline{\overline{a}}, \overline{\overline{a}}, \overline{\overline{a}}, \cdots\right)$ implies that $D < \overline{\overline{A}}$ in $\mathbf{F}^*$. The fact that all but finitely many of the coordinates of the diagonal tuple $\left(\overline{\overline{a}}_k, \overline{\overline{a}}_k, \overline{\overline{a}}_k, \cdots\right)$ are strictly less than the corresponding coordinate of $\left(\overline{\overline{a}}_0, \overline{\overline{a}}_1, \overline{\overline{a}}_2, \cdots\right)$ implies that $\overline{\overline{A}}_k < D$ in $\mathbf{F}^*$ for all k. Thus, the sublattice of $\mathbf{F}^*$ that is generated by $\{A, B, C, D\}$ contains elements ordered as in Figure 3. Let $E \in \mathrm{Con}(\mathbf{A})$ denote the join of the elements $\overline{\overline{A}}_k$, $k < \omega$. Observe that $E \leq D < \overline{\overline{A}}$. The proof of the theorem may be completed by proving the following claim.

Claim 3.9. *The elements* $\left\{\overline{\overline{A}}, \overline{\overline{A}} + B, \overline{\overline{A}} + C, B + C, B + E, C + E, E\right\}$ *consitute a sublattice of* $\mathbf{Con}(\mathbf{A})$ *that is isomorphic to* $\mathbf{D}_1$*. Hence* $\mathbf{D}_1 \in \mathcal{L}(\mathcal{V})$*.*

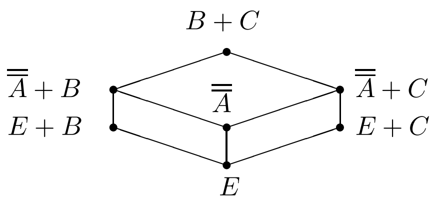

FIGURE 4. Some congruences of $\mathbf{A}$

This claim will be proved by applying Lemma 3.8 to the congruences $x := \overline{\overline{A}}$, $y := B + E$, and $z := C + E$. Using the fact that $\overline{\overline{A}} \geq E$, the statements (I)–(IV) from Lemma 3.8 that must be established may be written as:

(I) $\overline{\overline{A}} \leq (E + B) + (E + C)$,
(II) $(C + E)\left(\overline{\overline{A}} + B\right) \leq B + E$,
(III) $(B + E)\left(\overline{\overline{A}} + C\right) \leq C + E$, and
(IV) $\left(\overline{\overline{A}} + B\right)\left(\overline{\overline{A}} + C\right) \leq \overline{\overline{A}}$.

Item (I) is true since $B + C \geq \overline{\overline{A}}$. Item (IV) is true because $\overline{\overline{a}} = \left(\overline{\overline{a}} + b\right)\left(\overline{\overline{a}} + c\right)$ in **F**. Items (II) and (III) are symmetric, so we prove only (II). For this we have

$$\begin{aligned}
(C+E)\left(\overline{\overline{A}}+B\right) &= \left[(C+E)\left(C+\overline{\overline{A}}\right)\right]\left(\overline{\overline{A}}+B\right) \\
&= (C+E)\left[\left(C+\overline{\overline{A}}\right)\left(\overline{\overline{A}}+B\right)\right] \\
&= (C+E)\overline{\overline{A}} = \overline{\overline{A}}\left(C+\textstyle\sum_{k \text{ even}} \overline{\overline{A}}_k\right) \\
&= \overline{\overline{A}}\left(\textstyle\sum_{k \text{ even}} \left(C+\overline{\overline{A}}_k\right)\right) \\
&= \overline{\overline{A}}\left(\textstyle\sum_{k \text{ odd}} C_k\right) \\
&= \textstyle\sum_{k \text{ odd}} \overline{\overline{A}} C_k \qquad \text{(by the upper continuity of } \mathbf{Con}(\mathbf{A})) \\
&= \textstyle\sum_{k \text{ odd}} \overline{\overline{A}}_k = E \leq B+E.
\end{aligned}$$

To show that the sublattice generated by x, y and z is isomorphic to $\mathbf{D}_1$ we must show that $x \not\leq y$, i.e. $\overline{\overline{A}} \not\leq C+E$. If instead $\overline{\overline{A}} \leq C+E$, then from the middle lines of the previous calculation we would have $\overline{\overline{A}} = \overline{\overline{A}} \cdot \overline{\overline{A}} \leq \overline{\overline{A}}(C+E) = E$, contradicting our earlier conclusion that $E \leq D < \overline{\overline{A}}$. □

Next we turn to the proof of Theorem 3.2, which asserts that if $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and satisfies $\overline{\varepsilon}_m$ for some m, then it satisfies the stronger type of identity ε_M for some M. To make it easier to follow the argument, we briefly explain what properties the identities ε_m, $\overline{\varepsilon}_m$, and $\overline{\overline{\varepsilon}}_m$ express.

Given elements a, b, c in a lattice $\mathbf{L}$, the elements $b_k = \beta_k(a,b,c)$ and $a_k = a \cdot b_k$ for k even, $c_k = \gamma_k(a,b,c)$ and $a_k = a \cdot c_k$ for k odd, form a partial sublattice of $\mathbf{L}$, called a **herringbone**. This means that the unions of chains $\{a_k\}_{k=0}^{\infty} \cup \{b_{2k}\}_{k=0}^{\infty}$ and $\{a_k\}_{k=0}^{\infty} \cup \{c_{2k+1}\}_{k=0}^{\infty}$ are sublattices, that $a_{2k+1} + b_{2k} = b_{2k+2}$, $a_{2k+2} \cdot b_{2k} = a_{2k}$, $a_{2k+2} + c_{2k+1} = c_{2k+3}$, and $a_{2k+3} \cdot c_{2k+1} = a_{2k+1}$. The b_k's and c_k's lie in the interval

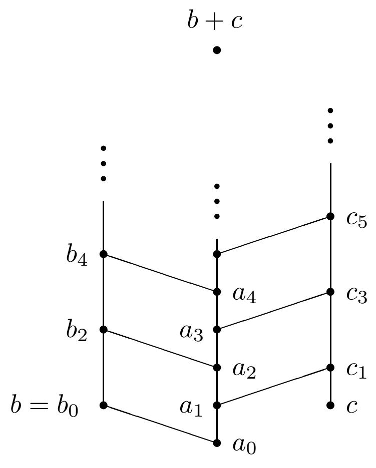

FIGURE 5.

$I[bc, b+c]$, which we call the **frame** of the herringbone. The element a is where the herringbone **starts**. If $b_{2k} = b_{2k+2}$ for some k, then it can be shown that $b_{2k} = b_{2\ell}$ for all $\ell \geq k$. When this happens we say that the herringbone **terminates after at most k steps**.

A lattice satisfies ε_{2m} if and only if all its herringbones terminate after at most m steps. A lattice satisfies $\overline{\varepsilon}_{2m}$ if and only if all herringbones that start at an element of the form $\overline{a} = a(b + c) + bc$ terminate after at most m steps. This barred element, $\overline{a}$, satisfies $bc \leq a(b + c) + bc = \overline{a} \leq b + c$, hence lies in the frame of the herringbone. Conversely if $u \in I[bc, b+c]$, then $\overline{u} = u(b+c)+bc = u$, so all elements in the frame are barred elements. Thus, $\overline{\varepsilon}_{2m}$ asserts that herringbones that start within the frame terminate after at most m steps. A lattice satisfies $\overline{\overline{\varepsilon}}_{2m}$ if and only if all herringbones that start at an element of the form $\overline{\overline{a}} = (a + b)(a + c)(b + c)$ terminate after at most m steps. This starting element, $\overline{\overline{a}}$, is the meet of an element $(a + b)(b + c)$ from the interval $I[b, b + c]$ and an element $(a + c)(b + c)$ from the interval $I[c, b + c]$. Call an element that is a meet of an element from $I[b, b + c]$ and an element from $I[c, b + c]$ a **product element** of the frame, so double barred elements are product elements. Conversely, if $u = vw$ is a product element with $v \in I[b, b+c]$ and $w \in I[c, b+c]$, then $u \leq \overline{\overline{u}} = (u+b)(u+c)(b+c) \leq vw(b+c) \leq u$, so the double barred elements are exactly the product elements. Thus, $\overline{\overline{\varepsilon}}_{2m}$ asserts that herringbones that start at a product element of the frame terminate after at most m steps.

In Theorem 3.1 we showed that if $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable and $\mathcal{V}$ satisfies an idempotent Maltsev condition, then there is an m such that all herringbones that start at product elements terminate after m steps. To prove Theorem 3.2 we will argue that if $\mathcal{L}(\mathcal{V})$ is first-order axiomatizable, satisfies $\overline{\overline{\varepsilon}}_m$ for some m, and there is no m such that all herringbones terminate after m steps, then there can be no M such that all herringbones that start within the frame terminate after M steps.

Proof of Theorem 3.2. We assume that $\mathcal{L}(\mathcal{V})$ is axiomatizable, $\mathcal{L}(\mathcal{V}) \models \overline{\overline{\varepsilon}}_n$ for some n, and that $\mathcal{L}(\mathcal{V}) \not\models \varepsilon_m$ for any m. We must show that $\mathcal{L}(\mathcal{V}) \not\models \overline{\varepsilon}_M$ for any M.

Let $\mathbf{F}$ be the lattice freely generated by $\{a, b, c\}$ relative to $\mathcal{L}(\mathcal{V})$. Choose a nonprincipal ultrafilter $\mathcal{U}$ on ω, and let $\mathbf{F}^*$ denote the ultrapower $\prod_{\mathcal{U}} \mathbf{F}$. Let $\Delta\colon \mathbf{F} \to \mathbf{F}^*$ be the diagonal embedding. As in the proof of Theorem 3.1 we may assume that $\mathbf{F}^*$ is a sublattice of $\mathbf{Con}(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{V}$.

Let $b_k = \beta_k(a, b, c)$, $c_k = \gamma_k(a, b, c)$, $a_k = a \cdot b_k$ if k is even and $a_k = a \cdot c_k$ if k is odd. Since $\mathbf{F}$ is free over $\{a, b, c\}$ in $\mathcal{L}(\mathcal{V})$, and we are assuming that identity ε_m fails in $\mathcal{L}(\mathcal{V})$ for all M, it follows that all b_k are distinct in $\mathbf{F}$. From the definition of the b_k, it follows that the a_k and the c_k are also distinct for all k. Apply the embedding Δ to these elements. Following the convention introduced in the proof of Theorem 3.1 we denote by upper case W the element $\Delta(w)$ for each $w \in F$. Now the B_k are congruences of $\mathbf{A}$, and $B_0 < B_2 < B_4 < \cdots$, just as in the proof of Theorem 3.1. Similarly $A_0 < A_1 < A_2 < \cdots$ and $C_1 < C_3 < C_5 < \cdots$. Thus, the herringbone in $\mathbf{Con}(\mathbf{A})$ with frame $I[BC, B + C]$ that starts at A does not terminate. To complete the proof we will show that the herringbone in $\mathbf{Con}(\mathbf{A})$ with frame $I[BC, B + C]$ that starts at the barred element $\overline{A} = A(B + C) + BC$ also does not terminate.

Claim 3.10. *The herringbone in* $\mathbf{Con}(\mathbf{A})$ *with frame* $I[BC, B+C]$ *that starts at* $A(B+C)$ *does not terminate.*

This claim holds because the herringbone with frame $I[BC, B+C]$ that starts at $A(B+C)$ is the same as the one that starts at A, which does not terminate by assumption.

Let $A_\omega = \sum_{k<\omega} A_k$, $B_\omega = B + A_\omega$ ($= B + \sum_{k<\omega} A_k = \sum_{k<\omega} B + A_k = \sum_{k \text{ even}} B_k$), and $C_\omega = C + A_\omega$ ($= \sum_{k \text{ odd}} C_k$), where the sum is taken in $\mathbf{Con}(\mathbf{A})$.

Claim 3.11. $\overline{A}B_\omega = \overline{A}C_\omega$

By the upper continuity of $\mathbf{Con}(\mathbf{A})$ we have

$$AB_\omega = A \sum_{k \text{ even}} B_k = \sum_{k \text{ even}} AB_k \leq \sum_{k \text{ odd}} C_k = C_\omega$$

and similarly $AC_\omega \leq B_\omega$, so in fact

$$AB_\omega = AC_\omega. \tag{3.1}$$

By (3.1) and the properties of the centralizer relation (Proposition 3.4 of [6] or Theorem 2.19 of [7]), $\mathbf{C}(B_\omega, A; AB_\omega)$ and $\mathbf{C}(C_\omega, A; AB_\omega)$, so $\mathbf{C}(B_\omega + C_\omega, A; AB_\omega)$, and therefore $A(B_\omega + C_\omega)$ is abelian over AB_ω. Since $B \leq B_\omega \leq B + C$ and $C \leq C_\omega \leq B + C$, we have $B_\omega + C_\omega = B + C$, so $A(B+C)$ is abelian over AB_ω. We will use the notation $\theta \lhd \psi$ to denote that $\theta \leq \psi$ and ψ is abelian over θ, so $P := AB_\omega \lhd A(B+C) =: Q$.

It follows from (3.1) that $AB_\omega \leq B_\omega C_\omega$. Since $\mathcal{V}$ satisfies a congruence identity, viz. $\overline{\overline{\varepsilon}}_n$, Corollary 4.12 of [9] guarantees that $\mathcal{V}$ has a weak difference term. But for varieties with a weak difference term, Lemma 6.10 of [7] guarantees that the relation $\lhd$ is compatible with join and meet, so

$$R := B_\omega C_\omega = AB_\omega + B_\omega C_\omega \lhd A(B+C) + B_\omega C_\omega =: S.$$

Part of this calculation shows that $R+Q = S$. We also have $RQ = B_\omega C_\omega A(B+C) = AB_\omega = P$, so $I[P,Q]$ and $I[R,S]$ are perspective abelian intervals. In this situation, the join map $x \mapsto x + R$ is a surjective function from $I[P,Q]$ to $I[R,S]$, according to Theorem 6.24 of [7]. Since $R = B_\omega C_\omega \leq B_\omega$ and $R \leq S$, the element $B_\omega S$ lies in $I[R,S]$, so there is an element $X \in I[P,Q]$ such that $X + R = B_\omega S$. We have

$$AB_\omega = P \leq X \leq (X+R)Q = (B_\omega S)(A(B+C)) \leq AB_\omega,$$

so $X = AB_\omega$. Therefore

$$B_\omega S = X + R = AB_\omega + B_\omega C_\omega = B_\omega C_\omega.$$

A similar argument show that $C_\omega S = B_\omega C_\omega$, so

$$B_\omega S = C_\omega S. \tag{3.2}$$

Now, since $B_\omega \geq B_0 = B$ and $C_\omega \geq C$, we have

$$S = A(B+C) + B_\omega C_\omega \geq A(B+C) + BC = \overline{A}, \tag{3.3}$$

hence $\overline{A}S = \overline{A}$. Combining the results of (3.2) and (3.3) we get $\overline{A}B_\omega = (\overline{A}S)B_\omega = \overline{A}(B_\omega S) = \overline{A}(C_\omega S) = \overline{A}C_\omega$, which yields the claim.

Claim 3.12. *There is a function $f\colon \omega \to \omega$ such that $\overline{A}B_n \leq C_{f(n)}$ and $\overline{A}C_n \leq B_{f(n)}$.*

Using Claim 3.11 we derive that

$$\overline{A}B_n \leq \overline{A}B_\omega = \overline{A}C_\omega \leq C_\omega. \tag{3.4}$$

If $D = (c_1, c_3, c_5, \ldots)/\mathcal{U} \in \mathbf{F}^* \leq \mathbf{Con}(\mathbf{A})$, then $C_k < D$ for all k in $\mathbf{Con}(\mathbf{A})$, since $(c_k, c_k, c_k, \ldots)$ is below $(c_1, c_3, c_5, \ldots)$ in all but finitely many coordinates. Therefore $C_\omega \leq D$ in $\mathbf{Con}(\mathbf{A})$. Together with (3.4) this yields that $\overline{A}B_n \leq D$ in $\mathbf{Con}(\mathbf{A})$. Since $\overline{A}B_n = (\overline{a}b_n, \overline{a}b_n, \ldots)/\mathcal{U}$, this means that $(\overline{a}b_n, \overline{a}b_n, \ldots)$ is below $(c_1, c_3, c_5, \ldots)$ in almost all coordinates modulo $\mathcal{U}$. Since the c_k's are increasing, this can only happen of $\overline{a}b_n \leq c_N$ for all sufficiently large N. Similarly it must be that for every n the inequality $\overline{a}c_n \leq b_N$ holds for all sufficiently large N. The claim follows by defining $f(n)$ to be the least natural number N for which both $\overline{a}b_n \leq c_N$ and $\overline{a}c_n \leq b_N$ hold.

Claim 3.13. *$\beta_k(\overline{A}, B, C) \leq B_{f^k(0)}$ and $\gamma_k(\overline{A}, B, C) \leq C_{f^k(0)}$.*

Claim 3.13 follows from Claim 3.12 by induction. For the basis of induction, $\beta_0(\overline{A}, B, C) = B = B_0 = B_{f^0(0)}$ and similarly $\gamma_0(\overline{A}, B, C) = C = C_0 = C_{f^0(0)}$. If the claim holds for some k, then

$$\begin{aligned}\beta_{k+1}(\overline{A}, B, C) &= B + \overline{A} \cdot \gamma_k(\overline{A}, B, C) \\ &\leq B + \overline{A}C_{f^k(0)} \\ &\leq B + B_{f^{k+1}(0)} \qquad \text{(Claim 3.12)} \\ &= B_{f^{k+1}(0)},\end{aligned}$$

and similarly $\gamma_{k+1}(\overline{A}, B, C) \leq C_{f^{k+1}(0)}$.

We now complete the proof of Theorem 3.2. Since $\overline{A} = A(B + C) + BC \geq A(B+C)$, the elements $\beta_k(\overline{A}, B, C)$ of the herringbone with frame $I[BC, B+C]$ that starts at $\overline{A}$ are term by term above the corresponding elements $\beta_k(A(B+C), B, C)$ of the herringbone with the same frame that starts at $A(B + C)$. But, as already noted in the proof of Claim 3.10, this latter herringbone is the same as one that starts at A. This and Claim 3.13 together show that $B_k \leq \beta_k(\overline{A}, B, C) \leq B_{f^k(0)}$. If the herringbone with frame $I[BC, B + C]$ starting at $\overline{A}$ terminated after at most k steps for some fixed k, then for all $\ell \geq k$ we would have

$$B_{2\ell} \leq \beta_{2\ell}(\overline{A}, B, C) = \beta_{2k}(\overline{A}, B, C) \leq B_{f^{2k}(0)},$$

forcing $B_{2\ell} \leq B_{f^{2k}(0)}$ for all $\ell \geq k$. This forces the herringbone starting at A to terminate, contrary to assumption. □

References

[1] G. Czédli, *Mal'cev conditions for Horn sentences with congruence permutability*, Acta Math. Hung. **44** (1984), 115–124.

[2] R. Freese and J. B. Nation, *Congruence lattices of semilattices*, Pacific J. Math. **49** (1973), 51–58.

[3] H.-Peter, Gumm, *Congruence modularity is permutability composed with distributivity*, Arch. Math. (Basel) **36** (1981), no. 6, 569–576

[4] J. Hagemann and A. Mitschke, *On n-permutable congruences*, Algebra Universalis **3** (1973), 8–12.
[5] C. Herrmann and W. Poguntke, *Axiomatic classes of modules*, Darmstadt Preprint **12**, 1972.
[6] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics, **76**, American Mathematical Society, 1988.
[7] K. A. Kearnes and E. W. Kiss, *The Shape of Congruence Lattices*, manuscript available at `http://spot.colorado.edu/~kearnes/Papers/cong.pdf`
[8] K. A. Kearnes, E. W. Kiss, M. A. Valeriote, *Minimal sets and varieties*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 1–41.
[9] K. A. Kearnes and Á. Szendrei, *The relationship between two commutators*, Internat. J. Algebra Comput. **8** (1998), 497-531.
[10] K. A. Kearnes and M. A. Valeriote, *A modification of Polin's variety*, Algebra Universalis **41** (1999), 229–231.
[11] P. Lipparini, *Congruence identities satisfied in n-permutable varieties*, Boll. Un. Mat. Ital. B (7) 8 (1994), no. 4, 851–868.
[12] P. Lipparini, *n–permutable varieties satisfy nontrivial congruence identities*, Algebra Universalis **33** (1995), 159–168.
[13] P. Lipparini, *A new proof that n-permutable varieties satisfy lattice identities*, manuscript available at `http://www.mat.uniroma2.it/~lipparin/nperm.dvi`
[14] P. Lipparini, *Every m-permutable variety satisfies the congruence identity $\alpha\beta_h = \alpha\gamma_h$*, to appear in Proc. Amer. Math. Soc.
[15] Á. Szendrei, *Strongly abelian minimal varieties*, Acta Sci. Math. (Szeged) **59** (1994), no. 1-2, 25–42.

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