ALMOST ALL MINIMAL IDEMPOTENT VARIETIES ARE CONGRUENCE MODULAR

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Abstract. We show that, up to term equivalence, the only minimal idempotent varieties that are not congruence modular are the variety of sets and the variety of semilattices. From this it follows that a minimal idempotent variety that is not congruence distributive is term equivalent to the variety of sets, the variety of semilattices, or a variety of affine modules over a simple ring.

1. Introduction

A variety $\mathcal{V}$ is *idempotent* if whenever $f(x_1, x_2, \dots, x_n)$ is a fundamental operation of $\mathcal{V}$, then $\mathcal{V} \models f(x, x, \dots, x) \approx x$. A variety is *minimal* if it is nontrivial and has no proper nontrivial subvariety.

Example 1.1. Let $\mathcal{V}$ be the variety of sets. This is the variety with no fundamental operations. $\mathcal{V}$ satisfies the definition of an idempotent variety by default. It is easy to see that $\mathcal{V}$ is minimal.

Example 1.2. Let $\mathcal{V}$ be the variety of semilattices. This is the variety defined with a single binary fundamental operation that is idempotent, commutative, and associative. It is known that $\mathcal{V}$ is minimal.

Example 1.3. Let $\mathbf{R}$ be a ring with unit. Let $\mathcal{V}$ be the variety of *affine* $\mathbf{R}$*-modules*: i.e., $\mathcal{V}$ consists of all reducts of $\mathbf{R}$–modules to the module operations of the form $r_1x_1 + \cdots + r_nx_n$ where $\Sigma r_i = 1$. $\mathcal{V}$ is an idempotent variety that is congruence modular and abelian, hence affine in the sense defined in [2]. $\mathcal{V}$ is minimal if and only if $\mathbf{R}$ is simple. Two varieties of this type are term equivalent if and only if the associated rings are isomorphic. Since there is a proper class of simple rings, there is a proper class of minimal idempotent affine varieties up to term equivalence.

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Example 1.4. Let κ be a cardinal and let $\mathbf{A}_\kappa$ be the algebra with universe κ whose set of fundamental operations is the set of all idempotent operations on κ. Let $\mathcal{V}_\kappa = \mathsf{HSP}(\mathbf{A}_\kappa)$ be the variety generated by $\mathbf{A}$. It is not difficult to show that $\mathcal{V}_\kappa$ is a congruence distributive minimal variety. Moreover if $\kappa \neq \lambda$, then $\mathcal{V}_\kappa$ and $\mathcal{V}_\lambda$ are not term equivalent since they have a different number of inequivalent binary terms. This shows that there is a proper class of congruence distributive minimal idempotent varieties up to term equivalence.

The purpose of this paper is to prove that Examples 1.1–1.3 constitute a complete list of the minimal idempotent varieties (up to term equivalence) that are not congruence distributive.

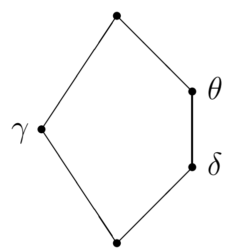
The study of minimal varieties has a long history which will not be recounted here. The reader is directed to [11] for a survey of results proved before 1992. Important new results on locally finite minimal varieties were obtained after that survey was written: see [5, 6, 12, 13], especially the discussion in the introduction of [6] that explains the relationship between these papers. The strong results for locally finite minimal varieties, which were proved using extensions of tame congruence theory, raise the question of whether anything similar is true for nonlocally finite minimal varieties. For locally finite minimal varieties we have a complete classification only for minimal *abelian* varieties, [5, 12, 13], and for minimal *idempotent* varieties, [9, 10]. It is not hard to construct pathological minimal abelian varieties (that have no locally finite analogue), but since about 1993 some have wondered if Szendrei's Theorem on locally finite, minimal, idempotent, varieties could be extended to nonlocally finite varieties. *Szendrei's Theorem* is the theorem in [10] that every locally finite, minimal, idempotent, variety is term equivalent to the variety of sets, the variety of semilattices, a variety of affine modules over a finite simple ring, or to one of a countable collection of congruence distributive varieties described in [9]. In 1993, I wrote the long paper [4] that proved a partial result in this direction for nonlocally finite varieties. Later, Szendrei and I proved in the last section of [7] a slightly stronger partial result. The reader will find in the current paper a very simple argument that establishes a version of Szendrei's Theorem for nonlocally finite varieties.

We still do not have a good description of the nonlocally finite, minimal, idempotent, congruence distributive varieties. As an example of how little we know, one may verify by consulting [9] that, excluding varieties generated by 2-element algebras, every locally finite, minimal, idempotent, congruence distributive variety is congruence 3-permutable and 3-distributive. So, it is possible that nonlocally finite, minimal, idempotent, congruence distributive varieties also must be congruence 3-permutable and 3-distributive. We have neither proofs nor counterexamples for either possibility.

2. The Proof

In Theorem 1 of [1], A. Day proves that if $\mathcal{V}$ is a variety and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(4)$, then $\mathrm{Con}\,(\mathbf{F})$ contains a "generic pentagon". That is, he exhibits five congruences on $\mathbf{F}$ and shows that if any algebra in $\mathcal{V}$ has a nonmodular congruence lattice, then the congruences he specifies must generate a pentagon in $\mathrm{Con}\,(\mathbf{F})$. The next lemma records the location of the generic pentagon.

Lemma 2.1. *Let* $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(a, u, v, b)$ *be a 4–generated free algebra in a variety* $\mathcal{V}$. *Let* $\theta = \mathrm{Cg}^{\mathbf{F}}((a, b), (u, v)), \gamma = \mathrm{Cg}^{\mathbf{F}}((a, u), (b, v))$, *and* $\delta = (\theta \wedge \gamma) \vee \mathrm{Cg}^{\mathbf{F}}(u, v)$. $\mathcal{V}$ *fails to be congruence modular if and only if* $(a, b) \notin \delta$.[1] *When this happens,* $\mathrm{Con}\,(\mathbf{F})$ *contains the following sublattice.*



$$(a, b) \in \theta - \delta$$

If θ is a congruence we will use the notation $a \equiv_\theta b$ as an alternative to $(a, b) \in \theta$.

Lemma 2.2. *Let* $\mathcal{V}$ *be a minimal idempotent variety. Let* $\mathbf{F}, \theta, \gamma$ *and* δ *be as in Lemma 2.1. Let* $\mathbf{G} = \mathrm{Sg}^{\mathbf{F}}(\{a, b\})$.

(1) *If* $\mathcal{V}$ *fails to be congruence modular, then* $\mathcal{V} = \mathsf{HSP}(\mathbf{G}/(\delta|_G))$.
(2) $G \times G \subseteq \theta \cap (\gamma \circ \delta \circ \gamma)$.

Proof. If $\mathcal{V}$ is not congruence modular, then according to Lemma 2.1 we have $(a, b) \notin \delta$. Thus, $\mathbf{G}/(\delta|_G)$ is a nontrivial algebra in $\mathcal{V}$. The minimality of $\mathcal{V}$ implies that $\mathbf{G}$ generates $\mathcal{V}$, so (1) holds.

[1] It is clear from the definitions that in any variety one has $\delta \leq \theta$, $\theta \wedge \gamma \leq \delta$, and $\theta \leq \gamma \vee \delta$. Thus, when $(a, b) \notin \delta$ it follows that $\delta < \theta$ and so γ, θ, and δ generate a pentagon in $\mathrm{Con}\,(\mathbf{F})$. What Day actually shows is that if $\mathrm{Con}\,(\mathbf{F})$ is modular, then it has no pentagon, so $(a, b) \in \delta$. From this he deduces that $\mathcal{V}$ satisfies a Mal'tsev condition strong enough guarantee congruence modularity. Thus, he has indeed proved that $\mathcal{V}$ fails to be congruence modular if and only if $(a, b) \notin \delta$.

Choose $(r, s) \in G \times G$. Since $\mathbf{G} = \mathrm{Sg}^{\mathbf{F}}(\{a, b\})$ there exist (idempotent) binary terms $r(x, y)$ and $s(x, y)$ such that $r = r(a, b)$ and $s = s(a, b)$ in $\mathbf{F}$. We have

$$r = r(a, b) \equiv_\theta r(a, a) = a = s(a, a) \equiv_\theta s(a, b) = s,$$

so $(r, s) \in \theta$. Since

$$\begin{aligned} r &= r(a, b) \equiv_\gamma r(u, v) \equiv_\delta r(u, u) = u \\ &= s(u, u) \equiv_\delta s(u, v) \equiv_\gamma s(a, b) = s, \end{aligned}$$

we have that $(r, s) \in \gamma \circ \delta \circ \gamma$. This proves (2). □

Lemma 2.3. *Let $\mathcal{V}$ be a minimal idempotent variety that is not congruence modular, and let t be a term in the language of $\mathcal{V}$. If $\mathcal{V} \models t(x, y, y) \approx x$, then $\mathcal{V} \models t(x, x, y) \approx x$.*

Proof. Let $\mathbf{F}, \theta, \gamma, \delta$ and $\mathbf{G}$ be as in Lemma 2.2. By Lemma 2.2 (1) it will suffice for us to prove that if $\mathcal{V} \models t(x, y, y) \approx x$, then $\mathbf{G}/(\delta|_G) \models t(x, x, y) \approx x$. This last condition means simply that in $\mathbf{F}$ we have

$$\forall r, s \in G \ \ (t(r, r, s) \equiv_\delta r).$$

So, assume that $\mathcal{V} \models t(x, y, y) \approx x$. Choose $r, s \in G$, and (by Lemma 2.2 (2)) find $p, q \in F$ such that $r \equiv_\gamma p \equiv_\delta q \equiv_\gamma s$. Calculating in $\mathbf{F}$,

$$r = t(r, p, p) \equiv_\delta t(r, p, q) \equiv_\gamma t(r, r, s).$$

But $t(r, r, s) \in G$, so by Lemma 2.2 (2) we have $r \equiv_\theta t(r, r, s)$. Thus,

$$t(r, r, s) \equiv_\theta r \equiv_\delta t(r, p, q).$$

Since $\delta \leq \theta$ we have $(t(r, r, s), t(r, p, q)) \in \theta \wedge \gamma \leq \delta$. This improves our earlier calculation to

$$r = t(r, p, p) \equiv_\delta t(r, p, q) \equiv_\delta t(r, r, s).$$

Since we have shown that $t(r, r, s) \equiv_\delta r$ for any $r, s \in G$ the proof is complete. □

Lemma 2.4. *Let $\mathcal{V}$ be a minimal idempotent variety that is not congruence modular. If $\mathcal{V} \models t(x, x, y) \approx x$ and $\mathcal{V} \models t(x, y, x) \approx x$, then $\mathcal{V} \models t(x, y, y) \approx x$.*

Proof. As in the last proof, it is enough to show that if $\mathcal{V} \models t(x, x, y) \approx x$ and $\mathcal{V} \models t(x, y, x) \approx x$, then in $\mathbf{F}$ we have

$$\forall r, s \in G \ \ (t(r, s, s) \equiv_\delta r).$$

Choose $r, s \in G$ and find $p, q \in F$ such that $r \equiv_\gamma p \equiv_\delta q \equiv_\gamma s$. The equation $t(x, x, y) \approx x$, which holds throughout $\mathcal{V}$, provides the equalities in

$$t(r, s, r) \equiv_\theta t(r, r, r) = r = t(r, r, p) \equiv_\theta t(r, s, p),$$

while the fact that $r \equiv_\gamma p$ implies that $t(r, s, r) \equiv_\gamma t(r, s, p)$. Thus, $(t(r, s, r), t(r, s, p)) \in \theta \wedge \gamma \leq \delta$. Similarly we have

$$t(r, s, q) \equiv_\theta t(r, r, q) = r = t(r, r, s) \equiv_\theta t(r, s, s),$$

while the fact that $q \equiv_\gamma s$ implies that $t(r, s, q) \equiv_\gamma t(r, s, s)$. Thus, $t(r, s, q) \equiv_\delta t(r, s, s)$. Putting the two conclusions together yields

$$t(r, s, r) \equiv_\delta t(r, s, p) \equiv_\delta t(r, s, q) \equiv_\delta t(r, s, s).$$

This conclusion and the equation $t(x, y, x) \approx x$ jointly imply that $t(r, s, s) \equiv_\delta r$ for any $r, s \in G$. The proof is finished. □

Lemma 2.5. *An idempotent variety $\mathcal{V}$ has no subvariety that is term equivalent to the variety of sets or to the variety of semilattices if and only if there is an $n > 1$, an n–ary term f of $\mathcal{V}$, and for every nonempty subset $K \subseteq N = \{1, \ldots, n\}$, there is an equation $f(x_{i_1}, \ldots, x_{i_n}) = f(y_{i_1}, \ldots, y_{i_n})$ satisfied in $\mathcal{V}$ where*

1. *all $x_{i_j}, y_{i_j} \in \{x, y\}$;*
2. *$\{x_{i_j} \mid j \in K\} = \{x\}$; and*
3. *$\{y_{i_j} \mid j \in K\} = \{y\}$ or $\{x, y\}$.*

Proof. This result is essentially Lemma 9.5 (1)⇔(3) of [3] in a different language. We explain why this is so.

Lemma 9.5 (1) of [3] is the condition that there is no clone homomorphism

$$\varphi : \mathrm{Clo}\,(\mathcal{V}) \longrightarrow \mathrm{Clo}\,(\mathcal{S})$$

from the clone of $\mathcal{V}$ to the clone of the variety $\mathcal{S}$ of semilattices. If we factor a potential homomorphism φ through its image, and use the fact that the only subclones of $\mathcal{S}$ are the full clone and the clone of projections, then we see that Lemma 9.5 (1) of [3] is exactly the condition that $\mathcal{V}$ has no clone homomorphism *onto* the clone of the variety of sets or the variety of semilattices. Equivalently, $\mathcal{V}$ has no subvariety term equivalent to the variety of sets or semilattices.

Lemma 9.5 (3) of [3] asserts that $\mathcal{V}$ has an n-ary term f such that for every nonempty subset $K \subseteq N = \{1, \ldots, n\}$, there is an equation $f(x_{i_1}, \ldots, x_{i_n}) = f(y_{i_1}, \ldots, y_{i_n})$ satisfied in $\mathcal{V}$ where $\{x_{i_j} \mid j \in K\} \neq \{y_{i_j} \mid j \in K\}$ and the x_{i_j} and y_{i_j} are variables. It is not explicitly stated that $\{x_{i_j} \mid j \in K\} = \{x\}$ and $\{y_{i_j} \mid j \in K\} = \{y\}$ or $\{x, y\}$, but it is easy to see that this can be arranged. In the equation

$f(x_{i_1}, \dots, x_{i_n}) = f(y_{i_1}, \dots, y_{i_n})$, in order to have $\{x_{i_j} \mid j \in K\} \neq \{y_{i_j} \mid j \in K\}$, there must be a variable that occurs in exactly one of these two sets. Substitute y for one such variable and substitute x for all other variables. This produces an equation involving only the variables x and y that is a consequence of the original equation and still implies that $\{x_{i_j} \mid j \in K\} \neq \{y_{i_j} \mid j \in K\}$. Thus, there is no loss of generality in assuming that all variables come from the set $\{x, y\}$. Moreover, one of these sets, which we may assume to be $\{x_{i_j} \mid j \in K\}$, must be $\{x\}$ while the other set must contain y. This shows that this lemma is merely a restatement of Lemma 9.5 (1)⇔(3). □

Theorem 2.6. *A minimal idempotent variety is term equivalent to the variety of sets, the variety semilattices, or is congruence modular.*

Proof. Let $\mathcal{V}$ be a minimal idempotent variety that is not congruence modular. We will assume that $\mathcal{V}$ is not term equivalent to the variety of sets or the variety of semilattices and derive a contradiction.

Since we are assuming that $\mathcal{V}$ is minimal and not term equivalent to the variety of sets or the variety of semilattices, we must have an operation $f(x_1, \dots, x_n)$ satisfying equations of the type described in Lemma 2.5. If $U \subseteq N = \{1, \dots, n\}$, then we will write f_U to denote the binary term that results from substituting x into f for x_i when $i \in U$ and substituting y for x_i when $i \notin U$. For example, $f_N = f(x, x, \dots, x)$ while $f_\emptyset = f(y, y, \dots, y)$. In this notation, the conditions listed in Lemma 2.5 express that for any nonempty $K \subseteq N$ there exist $V \supseteq K$ and $W \not\supseteq K$ such that $\mathcal{V} \models f_V \approx f_W$.

Let $\mathsf{F} = \{U \subseteq N \mid \mathcal{V} \models f_U \approx x\}$.

Claim 2.7. $N \in \mathsf{F}$. $\emptyset \notin \mathsf{F}$.

The statement that $N \in \mathsf{F}$ is just the statement that $\mathcal{V} \models f_N = f(x, x, \dots, x) \approx x$, that is a consequence of idempotence. But we cannot have $\mathcal{V} \models f_\emptyset = f(y, y, \dots, y) \approx x$, since with idempotence this would entail $\mathcal{V} \models y \approx x$. The assumption that $\mathcal{V}$ is a minimal variety excludes this.

Claim 2.8. *If $U \in \mathsf{F}$ and $V \supseteq U$, then $V \in \mathsf{F}$.*

Assume that the claim is false and fix $U \in \mathsf{F}$ and $V \supseteq U$ such that $V \notin \mathsf{F}$. Let $t(x, y, z)$ be the term obtained from $f(x_1, \dots, x_n)$ by substituting x for each $x_i, i \in U$, substituting y for each $x_i, i \in V - U$, and substituting z for each $x_i, i \notin V$. Since $U \in \mathsf{F}$ we have $\mathcal{V} \models t(x, y, y) = f_U \approx x$, while $V \notin \mathsf{F}$ implies that $\mathcal{V} \not\models t(x, x, y) = f_V \approx x$. This contradicts Lemma 2.3, since $\mathcal{V}$ is a minimal idempotent variety that is not congruence modular. The claim is proved.

Claim 2.9. *If $U, W \in \mathsf{F}$ and $U \cup W = N$, then $U \cap W \in \mathsf{F}$.*

Let $t(x, y, z)$ be the term obtained from f by substituting x for x_i when $i \in U \cap W$, substituting y for x_i when $i \in U - W$, and substituting z for x_i when $i \in W - U$. The assumption that $U \cup W = N$ guarantees that all variables are described by these cases. Since $U, W \in \mathsf{F}$ we have $\mathcal{V} \models t(x, x, y) \approx t(x, y, x) \approx x$. According to Lemma 2.4 this implies that $\mathcal{V} \models f_{U \cap W} = t(x, y, y) \approx x$, so $U \cap W \in \mathsf{F}$.

Now we complete the proof. Claims 2.7, 2.8 and 2.9 imply that F is nontrivial proper principal filter in the lattice of subsets of N. Thus, there is a set K with $\emptyset \neq K \neq N$ such that $\mathsf{F} = \{U \mid U \supseteq K\}$. As noted before Claim 2.7, Lemma 2.5 guarantees that there is a $V \supseteq K$ and a $W \not\supseteq K$ such that $\mathcal{V} \models f_V \approx f_W$. Comparing the definition of F to our conclusion that $\mathsf{F} = \{U \mid U \supseteq K\}$ we find that we have a contradiction:

$$\mathcal{V} \models f_V \approx x, \quad \mathcal{V} \not\models f_W \approx x \quad \text{and also} \quad \mathcal{V} \models f_V \approx f_W.$$

□

Corollary 2.10. *A minimal idempotent variety is term equivalent to the variety of sets, the variety semilattices, a variety of affine modules over a simple ring, or is congruence distributive.*

Proof. Assume that $\mathcal{V}$ is minimal, idempotent, and congruence modular. If $\mathcal{V}$ is not congruence distributive, then according to Exercise 8.1 of [2] some $\mathbf{A} \in \mathcal{V}$ has congruences α and β such that

$$\delta := [\alpha, \beta] < \alpha \wedge \beta =: \gamma.$$

Since the commutator is monotone in each variable, $[\gamma, \gamma] \leq [\alpha, \beta] = \delta$. Thus, γ/δ is a (nonzero) abelian congruence of $\mathbf{A}/\delta$. Since $\mathcal{V}$ is idempotent, any nontrivial class of γ/δ is a subuniverse that supports a nontrivial abelian subalgebra of $\mathbf{A}/\delta$. Therefore, if $\mathcal{V}$ is not congruence distributive, then it contains (and so is generated by) an abelian algebra. Thus, $\mathcal{V}$ is affine. It follows from the structure theorem for affine algebras (Proposition 2.6 of [8]) that an idempotent affine variety is term equivalent to a variety of affine modules. □

References

[1] A. Day, *A characterization of modularity for congruence lattices of algebras*, Canad. Math. Bull. **12** (1969), 167–173.

[2] R. Freese and R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, LMS Lecture Notes No. 125, Cambridge University Press, 1987.

[3] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics v. 76, American Mathematical Society, 1988.

[4] K. A. Kearnes, *Idempotent simple algebras*, in Logic and Algebra (Pontignano, 1994), 529–572, Lecture Notes in Pure and Appl. Math. **180**, 1996.

[5] K. A. Kearnes, E. W. Kiss and M. A. Valeriote, *Minimal sets and varieties*, Trans. Amer. Math. Soc. **350** (1998), 1–41.

[6] K. A. Kearnes and Á. Szendrei, *A characterization of minimal, locally finite varieties*, with Á. Szendrei, Trans. Amer. Math. Soc. **349** (1997), 1749–1768.

[7] K. A. Kearnes and Á. Szendrei, *The relationship between two commutators*, with Á. Szendrei, Internat. J. Algebra Comput. **8** (1998), 497–531.

[8] Á. Szendrei, Clones in Universal Algebra, Séminaire de Mathétiques Superieures v. 99, Les Presses de l'Université de Montréal, Montréal, 1986.

[9] Á. Szendrei, *Idempotent algebras with restrictions on subalgebras*, Acta Sci. Math. **51** (1987), 251–268.

[10] Á. Szendrei, *Every idempotent plain algebra generates a minimal variety*, Algebra Universalis, **25** (1988), 36–39.

[11] Á. Szendrei, *A survey on strictly simple algebras and minimal varieties*, in Universal Algebra and Quasigroup Theory, 209–239, Research and Exposition in Mathematics **19**, 1992.

[12] Á. Szendrei, *Strongly abelian minimal varieties*, Acta Sci. Math. (Szeged) **59** (1994), 25–42.

[13] Á. Szendrei, *Maximal non-affine reducts of simple affine algebras*, Algebra Universalis **34** (1995), 144–174.

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