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A Hamiltonian property for nilpotent algebras

K. A. Kearnes*

Abstract. In this paper we show that any finite algebra **A** satisfying a weak left nilpotence condition has the property that all maximal subuniverses are congruence blocks. Conversely, if every subalgebra of $\mathbf{A}^2$ has the property that all maximal subuniverses are congruence blocks, then **A** satisfies the aforementioned nilpotence condition.

1. Introduction

The algebra is said to be **Hamiltonian** if every nonempty subuniverse is a congruence block. This concept originates in group theory; every abelian group is ''Hamiltonian'' in the sense just defined, but Hamiltonian groups are usually defined to be *nonabelian* groups where every subgroup is a congruence block (i.e., is normal). For general algebras, the Hamiltonian property was fairly well understood (see [8]) before a satisfactory definition for the word ''abelian'' was settled on, so the general definition of ''Hamiltonian'' includes no restriction to nonabelian algebras. In fact, interest in the Hamiltonian property has been stimulated by the problem of determining its true relationship to the abelian property. We say that an algebra **A** is **abelian** if the diagonal subuniverse of $\mathbf{A}^2$ is a congruence block. Clearly this means that if $\mathbf{A}^2$ is Hamiltonian, then **A** is abelian. Conversely, it has been proven that if **A** is a finite algebra and every homomorphic image of a subalgebra of $\mathbf{A}^{|A|}$ is abelian, then **A** is Hamiltonian. See [7] for the proof.

Returning to groups, a well-known theorem of Wielandt is that a finite group is nilpotent if and only if its maximal subgroups are normal. (For a proof, see Corollary 10.3.4 of [1].) In this paper we will call an algebra **quasi-Hamiltonian** if its maximal subuniverses are congruence blocks. (Throughout, a **maximal subuniverse** will always mean a nonempty, proper subuniverse which is maximal under inclusion among all such subuniverses.) As we have stated it, Wielandt's Theorem does not immediately generalize to other types of algebras. For example, there are non-nilpotent *pointed*

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groups where every maximal subuniverse is a congruence block. (One can add a new constant operation to a carefully selected non-nilpotent group in such a way as to destroy all maximal subuniverses which are not congruence blocks.) However, there is a slight modification of Wielandt's Theorem which holds for pointed groups:

THEOREM. *Let* **A** *be a finite pointed group. If* **A** *is nilpotent, then* **A** *is quasi-Hamiltonian. Conversely, if every subalgebra of* $\mathbf{A}^2$ *is quasi-Hamiltonian, then* **A** *is nilpotent.*

In fact, this version of Wielandt's Theorem holds not just for finite pointed groups, but for any finite algebra which generates a congruence modular variety. This fact can be deduced from Corollary 3.8 of this paper. In this paper, we generalize Wielandt's Theorem to any finite algebra. This requires isolating the correct notion of nilpotence.

A commutator theory for congruences of arbitrary algebras is outlined in Chapter 3 of [2]. An algebra is said to be **left nilpotent** if for some $k > 0$ it is the case that

$$(1]^{k+1} := [1, \ldots [1, [1, 1]]] = 0, \qquad (k \text{ pairs of brackets}).$$

The definition of right nilpotence is symmetric to this one. Nilpotence properties of finite algebras are studied in [3] and there it is shown that the properties

L. **A** is left nilpotent.
HL. Every homomorphic image of **A** is left nilpotent
R. **A** is right nilpotent.
HR. Every homomorphic image of **A** is right nilpotent

are related as in the diagram

$$\begin{array}{ccc} HR & \Rightarrow & HL \\ \Downarrow & & \Downarrow \\ R & \Rightarrow & L \end{array}$$

In particular, left nilpotence is the weakest form of nilpotence one might express in terms of the commutator. However, there is a slightly weaker nilpotence condition that can be phrased in terms of tame congruence theory:

$$C(1, N^2; \delta) \text{ holds whenever } \delta \prec \theta \text{ and } N \text{ is a } \langle \delta, \theta \rangle\text{-trace.} \qquad (\dagger)$$

We prove in this paper that (†) is weaker than left nilpotence. Our main result is the following extension of Wielandt's Theorem: If $\mathbf{A}$ is a finite algebra satisfying (†), then $\mathbf{A}$ is quasi-Hamiltonian. Conversely, if every subalgebra of $\mathbf{A}^2$ is quasi-Hamiltonian, then $\mathbf{A}$ satisfies (†). It is not hard to show that, in a congruence modular variety, condition (†) is equivalent to the four previously displayed nilpotence conditions and for such varieties these conditions describe what is usually referred to simply as ''nilpotence''.

The version of Wielandt's Theorem which we prove here has an interesting application. The paper [6] characterizes minimal locally finite varieties via a Mal'cev-like condition. For varieties generated by abelian algebras, this result from [6] is weaker than the results in [5] or the results combined in [12] and [13]. But one can recover these stronger results from the Mal'cev-like condition of [6] by applying our version of Wielandt's Theorem. The way to do this is explained in Section 4 of [6]. We also mention that Theorem 2.1 of this paper, which describes how minimal abelian congruences restrict to maximal subalgebras, is interesting in its own right. This theorem is critical in the proof of the main result of [4] which describes when a locally finite variety has a locally solvable direct factor. In particular, [4] uses Theorem 2.1 to simplify the procedure for proving that a locally finite variety is of the form $\mathscr{V}_1 \times \mathscr{V}_2 \times \mathscr{V}_3$ where $\mathscr{V}_1$ is locally strongly solvable, $\mathscr{V}_2$ is locally solvable and congruence permutable, and $\mathscr{V}_3$ is neutral. One can substantially shorten the McKenzie-Valeriote decidability proof in [11] using this descendant of Theorem 2.1.

Our reference for notation and general algebraic information is [10], our reference for tame congruence theory is [2] and our reference for commutator theory is both [2] and [3].

2. A preliminary result

In this section, we examine how minimal abelian congruences on finite algebras restrict to subalgebras. The earliest ancestor of this result is the title result of M. Valeriote's paper, [14]. Section 4 of R. McKenzie's paper, [9], contains an evolved version. We further refine the arguments to obtain the following:

THEOREM 2.1. *Assume that* $\mathbf{A}$ *is a finite algebra, B is a maximal subuniverse of* $\mathbf{A}$ *and γ is a minimal abelian congruence in* $\operatorname{Con}\mathbf{A}$. *If* $\gamma|_B > 0$, *then B is a union of γ-blocks.*

We shall argue by contradiction to establish this theorem. So, in addition to the hypotheses of the theorem, we shall assume that $\gamma|_B > 0$ and that B is *not* a union of γ-blocks. In this paper we shall use the notation that if $S \subseteq A$ and $\theta \in \mathrm{Con}\,\mathbf{A}$, then $S^\theta = \{x \in A \mid (x, y) \in \theta \text{ for some } y \in S\}$. Note that B^γ is a subuniverse of $\mathbf{A}$ which contains B and is a union of γ-blocks. By the maximality of B, our assumption that B is not a union of γ-blocks may be expressed as: $B^\gamma = A$.

There is no effect to either the hypotheses or conclusion of Theorem 2.1 if we expand $\mathbf{A}$ by adding constant operations for each element of B. This has the effect of simplifying some notation, so we assume in our proof that every element of B is the interpretation of a constant term. This implies, in particular, that $\mathbf{A}$ is generated by any $a \in A - B$. For any such a and any unary polynomial $p \in \mathrm{Pol}_1\mathbf{A}$ we may always express p as $p(x) = t^{\mathbf{A}}(x, a)$ where t is some binary term.

LEMMA 2.2. *There exists a $U \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$ and an idempotent term operation $e(x)$ such that $e(A) = U$ and the body of U is contained in B. Furthermore, if $V \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$ and $V = h(A)$ for some term operation h satisfying $h(\gamma) \nsubseteq 0_{\mathbf{A}}$, then the body of V is contained in B.*

Proof. First we will show that there is a $\langle 0, \gamma \rangle$-trace of $\mathbf{A}$ which contains at least two elements of B. Assume instead that none of the $\langle 0, \gamma \rangle$-traces contain two elements of B. Since $\gamma|_B > 0$, there exists a pair $(c, d) \in \gamma|_B - 0_B$. The γ-block in $\mathbf{A}$ containing these elements is connected by traces. Let N be a $\langle 0, \gamma \rangle$-trace containing c. Our assumption that no two elements of a trace lie in B forces $N \cap B = \{c\}$. There is a unary polynomial f such that $f(A) \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$, $f(c) \neq f(d)$ and $\{f(c), f(d)\} \subseteq N$. If $N = \{f(c), f(d)\}$, then certainly we must be in one of the following cases:

Case I. $f(c) = c$, or

Case II. $f(d) = c$.

If we are in neither case, then N contains the three distinct elements c, $f(c)$ and $f(d)$. If this happens, then the monoid of unary polynomials of $\mathbf{A}|_N$ contains a group of permutations acting transitively on N. Composing one of these with f if necessary one can construct a polynomial f' for which $f'(A) = f(A)$, $f'(c) = c \neq f'(d)$ and $\{f'(c), f'(d)\} \subseteq N$. Replacing f with f' if necessary, we may assume that f has been chosen so that we are in either Case I or Case II.

If $f(c) = c$, then define $a = f(d) \in N - \{c\}$. Since $a \in A - B$, we may express f as $f(x) = t^{\mathbf{A}}(x, a)$ where $t(x, y)$ is a term. By assumption, no $\langle 0, \gamma \rangle$-trace contains two distinct elements of B, so no polynomial image of a $\langle 0, \gamma \rangle$-trace contains two distinct elements of B. Because $t^{\mathbf{A}}(c, N)$ contains the elements $t^{\mathbf{A}}(c, a) = c \in B$ and $t^{\mathbf{A}}(c, c) \in B$, we must have

$$t^{\mathbf{A}}(c, a) = c = t^{\mathbf{A}}(c, c).$$

Since γ is abelian this implies that

$$a = t^{\mathbf{A}}(d, a) = t^{\mathbf{A}}(d, c) \in B,$$

which is false. This takes care of Case I. Next, assume that f can be chosen so that $f(d) = c$. Now define a by $a = f(c)$ $(\neq f(d) = c)$. Since $N \cap B = \{c\}$, we again have that $a \in A - B$. Hence we may express $f(x) = t^{\mathbf{A}}(x, a)$ again. Arguing as before, $t^{\mathbf{A}}(d, N)$ contains at most one element of B and that element must be $c = t^{\mathbf{A}}(d, a)$ and also $t^{\mathbf{A}}(d, c)$. Hence

$$t^{\mathbf{A}}(d, a) = t^{\mathbf{A}}(d, c),$$

which forces the contradiction

$$a = t^{\mathbf{A}}(c, a) = t^{\mathbf{A}}(c, c) \in B.$$

We cannot be in Case I or Case II and so, as these are the only cases, there must be a $\langle 0, \gamma \rangle$-trace containing two elements of B.

Let N be a $\langle 0, \gamma \rangle$-trace containing the distinct elements $u', v' \in B$. Let $e'(x)$ be an idempotent polynomial of $\mathbf{A}$ for which $N \subseteq e'(A) \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$. Choose $(a', b') \in ((A - B) \times B) \cap \gamma$. We may express e' as $s^{\mathbf{A}}(x, a')$, since $a' \in A - B$, and furthermore we may do so with a term s where $\mathbf{A} \models s(s(x, y), y) = s(x, y)$. Since

$$s^{\mathbf{A}}(u', a') = u' \neq v' = s^{\mathbf{A}}(v', a')$$

and $(u', v'), (a', b') \in \gamma$, we get that

$$s^{\mathbf{A}}(u', b') \neq s^{\mathbf{A}}(v', b').$$

Hence $s^{\mathbf{A}}(\gamma, b') \nsubseteq 0_{\mathbf{A}}$ which implies that $s^{\mathbf{A}}(A, b')$ contains a $\langle 0, \gamma \rangle$-minimal set.

The fact that γ is abelian implies that every prime quotient of $\mathbf{A}$ is γ-coherent. (This follows from an application of Theorem 4.20, Lemma 4.13 and Lemma 4.2 of [3].) From this and Theorem 4.3 of [3] we get that, whenever $\mathbf{A} \models s(s(x, y), y) = s(x, y)$ and $(a', b') \in \gamma$, we can conclude that the polynomials $s^{\mathbf{A}}(x, a')$ and $s^{\mathbf{A}}(x, b')$ have ranges of the same size. But $s^{\mathbf{A}}(A, a') = e'(A) \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$ and $s^{\mathbf{A}}(A, b')$ contains a $\langle 0, \gamma \rangle$-minimal set. Since all $\langle 0, \gamma \rangle$-minimal sets have the same size, we conclude that $s^{\mathbf{A}}(A, b') \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$. Set $U = s^{\mathbf{A}}(A, b')$. U is a $\langle 0, \gamma \rangle$-minimal set which is the range of the idempotent term operation, $s^{\mathbf{A}}(x, b')$. As argued in the previous paragraph, $u = s^{\mathbf{A}}(u', b') \neq s^{\mathbf{A}}(v', b') = v$ are distinct, γ-related elements of $U \cap B$. Let M denote the trace of U which contains these elements and let $e(x) = s^{\mathbf{A}}(x, b')$. Our first goal will be to show that $M \subseteq B$. Then we shall argue that all other traces in U belong to B as well. This will establish the first assertion of the lemma. From this point on, the proof of this lemma closely parallels McKenzie's arguments from Section 4 of [9].

Case 1. $\mathrm{typ}(0, \gamma) = \mathbf{1}$.

Proof for Case 1. Suppose that $M \nsubseteq B$. Choose $a \in M - B$. Since $0 \prec \gamma$, $\mathbf{A}|_M$ is simple. There is a polynomial $g \in \mathrm{Pol}_1 \mathbf{A}|_M$ such that, say, $eg(u) = a \neq eg(v)$. If $eg(x) = r^{\mathbf{A}}(x, a)$, then we have $r^{\mathbf{A}}(x, y) \in \mathrm{Pol}_2 \mathbf{A}|_M$, so $r^{\mathbf{A}}(x, y)$ depends on only one variable on M. Since $r^{\mathbf{A}}(u, a) \neq r^{\mathbf{A}}(v, a)$, $r^{\mathbf{A}}(x, y)$ depends on only its first variable. Hence

$$a = r^{\mathbf{A}}(u, a) = r^{\mathbf{A}}(u, u) \in B,$$

a contradiction, so $M \subseteq B$.

Now let M' be any other trace in U. Choose some element $a' \in M' - B$ if possible. Since $B^{\gamma} = A$ we can choose an element $b' \in B$ with $(a', b') \in \gamma$. By applying e if necessary, we may assume that $b' \in U$, so in fact $b' \in M'$. All traces are polynomially isomorphic, so we can find a polynomial $ep(x)$ such that $ep(M) = M'$. If we write $ep(x)$ as $q^{\mathbf{A}}(x, a')$, then we can conclude that $q^{\mathbf{A}}(M, M') = M'$. The operation $q^{\mathbf{A}}(x, y)$ restricted to $M \times M'$ depends on only one variable, which certainly must be the first, so

$$M' = q^{\mathbf{A}}(M, a') = q^{\mathbf{A}}(M, b') \subseteq B.$$

This shows that all traces of U belong to B, so the body of U is contained in B.

Case 2. $\mathrm{typ}(0, \gamma) = \mathbf{2}$.

Proof for Case 2. Suppose that $M \nsubseteq B$. As in Case 1, we begin by choosing $a \in M - B$. Let $d(x, y, z)$ be a pseudo-Mal'cev polynomial of U. We may write $ed(x, y, z) = h^{\mathbf{A}}(x, y, z, a)$ for some term h. From the properties of a pseudo-Mal'cev operation, the polynomials $h^{\mathbf{A}}(x, u, u, a)$ and $h^{\mathbf{A}}(u, x, u, a)$ are permutations of U. Since γ is abelian and $u, a \in M$, the term operation $k(x, y) = h(x, y, u, u)$ has the property that $k^{\mathbf{A}}(x, u)$ and $k^{\mathbf{A}}(u, x)$ are permutations of M. It follows that $k^{\mathbf{A}}(x, y)$ is a quasigroup operation on M. The argument of Lemma 4.6 of [2] explains how to construct a term from k which interprets as a Mal'cev operation on M. The argument of Lemma 4.20 of [2] explains how to alter this term to one which interprets as a pseudo-Mal'cev operation on U. Hence we may assume that $d(x, y, z)$ is a term operation of $\mathbf{A}$.

The algebra $\mathbf{A}|_M$ is polynomially equivalent to a 1-dimensional vector space. We plan to show that the vector space operations of $\mathbf{A}|_M$ are the restrictions to M of term operations. We already know that the Mal'cev operation $x - y + z$ of $\mathbf{A}|_M$ is given by the restriction to M of $d^{\mathbf{A}}(x, y, z)$ and that at least two constants are given by term operations since $|M \cap B| \geq 2$. It will be enough to show that all the scalar multipliers are given by term operations. Following the argument in [9] we use a counting argument. Assume that the unary polynomials of $\mathbf{A}|_M$ which are permutations fixing some chosen element $0 \in M \cap B$ are given by $\{g_1(x), \ldots, g_n(x)\}$. We can write each $eg_i(x)$ as $t_i^{\mathbf{A}}(x, a)$ for terms t_i. The set $\{t_1^{\mathbf{A}}(x, 0), \ldots, t_n^{\mathbf{A}}(x, 0)\}$ is a set of unary term operations which are 1-1 on M. Since U is closed under each $t_i^{\mathbf{A}}(x, 0)$ and $t_i^{\mathbf{A}}(u, a)\ \gamma\ t_i^{\mathbf{A}}(u, 0)$, it follows that M is closed under each $t_i^{\mathbf{A}}(x, 0)$ and that these term operations are permutations of M. We write $x - y$ for the term operation $d^{\mathbf{A}}(x, y, 0)$. Let $G_i(x) = t_i(x, 0) - t_i(0, 0)$. Each $G_i^{\mathbf{A}}(x)$ is a unary term operation which fixes 0 and is a permutation of M; hence

$$\{G_1^{\mathbf{A}}(x), \ldots, G_n^{\mathbf{A}}(x)\} \subseteq \{g_1(x), \ldots, g_n(x)\}.$$

To show that all unary multipliers are given by term operations it suffices to show that

$$G_i^{\mathbf{A}} = G_j^{\mathbf{A}} \quad \Rightarrow \quad g_i = g_j.$$

For this we must prove that

$$t_i^{\mathbf{A}}(x, 0) - t_i^{\mathbf{A}}(0, 0) = t_j^{\mathbf{A}}(x, 0) - t_j^{\mathbf{A}}(0, 0)$$

on M only if $t_i^{\mathbf{A}}(x, a) = t_j^{\mathbf{A}}(x, a)$ on M. We rewrite the previous displayed line as

$$t_i^{\mathbf{A}}(x, 0) - t_j^{\mathbf{A}}(x, 0) = t_i^{\mathbf{A}}(0, 0) - t_j^{\mathbf{A}}(0, 0).$$

Now we use the fact that γ is abelian to change some 0's to a's:

$$t_i^{\mathbf{A}}(x, a) - t_j^{\mathbf{A}}(x, a) = t_i^{\mathbf{A}}(0, a) - t_j^{\mathbf{A}}(0, a) = 0 - 0 = 0.$$

Hence $t_i^{\mathbf{A}}(x, a) = t_j^{\mathbf{A}}(x, a)$, as we hoped. It follows that all the vector space operations of $\mathbf{A}|_M$ are the restrictions to M of term operations.

The set $M \cap B$ is a subuniverse of the 1-dimensional vector space structure on M and this set contains at least two elements. It follows that $M \cap B = M$ or, equivalently, $M \subseteq B$.

Now let M' be any other trace in U. Choose some element $a' \in M' - B$ if possible. As in Case 1, we can find some $b' \in M' \cap B$. Choose some element $c \in M$ and consider the unary term operation $d^{\mathbf{A}}(b', c, x)$. This term operation is a permutation of U, by the properties of a pseudo-Mal'cev operation, and it maps M onto M'. This is impossible if $M \subseteq B$ and $M' \nsubseteq B$, since B is closed under unary term operations. We conclude that the body of U is contained in B.

To finish the proof of the lemma we need to prove that if $V \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$ and $V = h(A)$ for some term operation h satisfying $h(\gamma) \nsubseteq 0_{\mathbf{A}}$, then the body of V is contained in B. By Theorem 2.8 of [2] there is a polynomial operation $q(x)$ such that $hq|_U: U \to V$ is a bijection. Choose any pair $(a, b) \in ((A - B) \times B) \cap \gamma$ and write $q(x) = t^{\mathbf{A}}(x, a)$ for some term t. The polynomial $ht^{\mathbf{A}}(x, a)$ satisfies $ht^{\mathbf{A}}(\gamma|_U, a) \nsubseteq 0_{\mathbf{A}}$, so $ht^{\mathbf{A}}(\gamma|_U, b) \nsubseteq 0_{\mathbf{A}}$, since γ is abelian. Hence $ht^{\mathbf{A}}(x, b)$ maps U onto a minimal set. But $ht^{\mathbf{A}}(U, b)$ is contained in the minimal set $V = h(A)$. It follows that $ht^{\mathbf{A}}(x, b)$ maps U onto V. Since it is a bijection, $ht^{\mathbf{A}}(x, b)$ maps the body of U onto the body of V. But $ht^{\mathbf{A}}(x, b)$ is a term operation, so it maps B into B. We conclude that

$$\text{body of } V = ht^{\mathbf{A}}(\text{body of } U, b) \subseteq B.$$

This completes the proof. □

Proof of Theorem 2.1. Assume that $\mathbf{A}$ is a finite algebra, B is a maximal subuniverse of $\mathbf{A}$ and γ is a minimal abelian congruence in Con $\mathbf{A}$. Assume also that $\gamma|_B > 0$ and that $B^{\gamma} = A$. We will derive a contradiction from these assumptions.

Since $B^{\gamma} = A$, there exists a pair $(a, b) \in ((A - B) \times B) \cap \gamma$. Each γ-block is connected by traces, so we can even choose a and b so that they come from some $\langle 0, \gamma \rangle$-trace, N. Let $R = \{a\}^{\gamma} \cap B$. Choose $W \in \mathrm{M}_{\mathbf{A}}(0, \gamma)$ with $N \subseteq W$ and choose an idempotent unary polynomial $e(x)$ such that $e(A) = W$. We may express e as $e(x) = s^{\mathbf{A}}(x, a)$ where $\mathbf{A} \models s(s(x, y), y) = s(x, y)$. Since $s^{\mathbf{A}}(x, a) = x$ on W, we have $s^{\mathbf{A}}(a, a) = a$ and $s^{\mathbf{A}}(b, a) = b$. As we argued in the proof of Lemma 2.2, the fact that γ is abelian implies that for any $c \in R$ it is the case that $U_c := s^{\mathbf{A}}(A, c)$ is a $\langle 0, \gamma \rangle$-minimal set. The γ, γ-term condition implies that, since $s^{\mathbf{A}}(\gamma|_W, a) \nsubseteq 0_{\mathbf{A}}$ holds, it is the case that $s^{\mathbf{A}}(\gamma|_W, c) \nsubseteq 0_{\mathbf{A}}$, too. It follows that $s^{\mathbf{A}}(x, c)$ is a polynomial isomorphism of W onto U_c. Thus,

$$s^{\mathbf{A}}(\text{body of } W, c) = \text{body of } U_c.$$

Since U_c is the image of the idempotent term operation $s^{\mathbf{A}}(x, c)$, Lemma 2.2 ensures that the body of U_c is contained in B. For any $c \in R$ we get that

$$s^{\mathbf{A}}(a, c) \in s^{\mathbf{A}}(\text{body of } W, c) = \text{body of } U_c \subseteq B.$$

Furthermore, $s^{\mathbf{A}}(a, c)\,\gamma\, s^{\mathbf{A}}(a, a) = a$. Hence $s^{\mathbf{A}}(a, R) \subseteq R$. There exists a $k > 0$ such that, for $t(x, y) = s^{k}_{(1)}(x, y)$ $(:= s(x, s(x, s(x, \ldots, s(x, y))\ldots)))$, we have $\mathbf{A} \models t(x, t(x, y)) = t(x, y)$. Of course, the fact that $s^{\mathbf{A}}(a, R) \subseteq R$ implies that $t^{\mathbf{A}}(a, R) \subseteq R$. We also have $t^{\mathbf{A}}(a, a) = a$ and $t^{\mathbf{A}}(b, a) \in R$. Applying the γ, γ-term condition to

$$t^{\mathbf{A}}(b, a) = t^{\mathbf{A}}(b, t^{\mathbf{A}}(b, a))$$

we get that

$$a = t^{\mathbf{A}}(a, a) = t^{\mathbf{A}}(a, t^{\mathbf{A}}(b, a)) \in t^{\mathbf{A}}(a, R) \subseteq R.$$

But $R \subseteq B$ and $a \notin B$. This contradiction finishes the proof. □

3. Quasi-Hamiltonian algebras

In his section, we look at quasi-Hamiltonian algebras and **quasi-Hamiltonian varieties** (defined to be the varieties consisting of quasi-Hamiltonian algebras). In order to say when a variety consists of quasi-Hamiltonian algebras, we need to describe the exact equational property of a (locally finite) variety which is associated with being quasi-Hamiltonian. For this, we introduce the following definition.

DEFINITION 3.1. Let $\mathbf{A}$ be an algebra. Polynomials $f(x), g(x) \in \mathrm{Pol}_1 \mathbf{A}$ are said to be **twins** if there is a term $t(x, \bar{y})$ and $\bar{a}, \bar{b} \in A^n$ such that $f(x) = t^{\mathbf{A}}(x, \bar{a})$ and $g(x) = t^{\mathbf{A}}(x, \bar{b})$.

Concerning the notation of the previous definition, we often write $t^{\mathbf{A}}_{\bar{a}}(x)$ and $t^{\mathbf{A}}_{\bar{b}}(x)$ in place of $t^{\mathbf{A}}(x, \bar{a})$ and $t^{\mathbf{A}}(x, \bar{b})$ in order to emphasize that these polynomials are thought of as functions of x.

If A is a finite set of size m and $t: A \to A$ is a function, then $e(x) := t^{m!}(x)$ is an idempotent function on A. Hence, if $t_{\bar{y}}(x) = t(x, \bar{y})$ is an $(n+1)$-ary term of the algebra $\mathbf{A}$, where $|A| = m$, then $e^{\mathbf{A}}_{\bar{y}}(x) := (t^{\mathbf{A}}_{\bar{y}})^{m!}(x)$ is idempotent as a function of x for each fixed choice of values for $\bar{y}$. That is,

$$\mathbf{A} \models e_{\bar{y}}(e_{\bar{y}}(x)) = e_{\bar{y}}(x).$$

In this case, for any $\bar{a} \in A^n$, the unary polynomial $e^{\mathbf{A}}_{\bar{a}}(x)$ is idempotent. Conversely, if $f(x) = s^{\mathbf{A}}(x, \bar{b}) = s^{\mathbf{A}}_{\bar{b}}(x)$ is an idempotent polynomial for some term s and some tuple $\bar{b} \in A^l$, then by replacing $s_{\bar{y}}(x)$ with $r_{\bar{y}}(x) := (s_{\bar{y}})^{m!}(x)$ we get that $f(x) = r_{\bar{b}}^{\mathbf{A}}(x)$ where

$$\mathbf{A} \models r_{\bar{y}}(r_{\bar{y}}(x)) = r_{\bar{y}}(x).$$

It follows that if $t_{\bar{a}}^{\mathbf{A}}(x)$ and $t_{\bar{b}}^{\mathbf{A}}(x)$ are arbitrary idempotent twin polynomials of $\mathbf{A}$, then replacing $t_{\bar{y}}(x)$ with an iterate if necessary, we may assume that

$$\mathbf{A} \models t_{\bar{y}}(t_{\bar{y}}(x)) = t_{\bar{y}}(x).$$

In this section, we shall be interested in the property:

Idempotent twins have ranges of the same size. (‡)

For a finite algebra $\mathbf{A}$, (‡) is an equational property. For, assume that $|A| = m$ and that $t(x, \bar{y})$ is an arbitrary term. Let $e_{\bar{y}}(x) = (t_{\bar{y}})^{m!}(x)$. If $\bar{a}, \bar{b} \in A^n$, then $e_{\bar{a}}^{\mathbf{A}}(x)$ and $e_{\bar{b}}^{\mathbf{A}}(x)$ are a typical pair of idempotent twins. The equation

$$(e_{\bar{a}}^{\mathbf{A}} \circ e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{m!}(x) = e_{\bar{a}}^{\mathbf{A}}(x)$$

must hold if the idempotent twins $(e_{\bar{a}}^{\mathbf{A}} \circ e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{m!}(x)$ and $(e_{\bar{a}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{m!}(x) = e_{\bar{a}}^{\mathbf{A}}(x)$ have ranges of equal size. Conversely, if this equation holds, then $|e_{\bar{a}}^{\mathbf{A}}(A)| \leq |e_{\bar{b}}^{\mathbf{A}}(A)|$; therefore a similar equation with $\bar{a}$ and $\bar{b}$ interchanged will imply that $e_{\bar{a}}^{\mathbf{A}}$ and $e_{\bar{b}}^{\mathbf{A}}$ have ranges of the same size. Since $\bar{a}, \bar{b} \in A^n$ were chosen arbitrarily, (‡) implies that $\mathbf{A}$ must satisfy all equations of the form

$$((t_{\bar{y}})^{m!} \circ (t_{\bar{z}})^{m!} \circ (t_{\bar{y}})^{m!})^{m!}(x) = (t_{\bar{y}})^{m!}(x).$$

Conversely, the satisfaction of all such equations implies that $\mathbf{A}$ satisfies (‡).

Saying that the finite algebras in a locally finite variety $\mathscr{V}$ satisfies (‡) is also equational. The correct property is that for every $(n+1)$-ary term $t(x, \bar{y})$ there should exist a positive integer K such that $((t_{\bar{y}})^K \circ (t_{\bar{y}})^K)(x) = (t_{\bar{y}})^K(x)$ and

$$((t_{\bar{y}})^K \circ (t_{\bar{z}})^K \circ (t_{\bar{y}})^K)^K(x) = (t_{\bar{y}})^K(x)$$

are equations of $\mathscr{V}$. The choice $K = |F_{\mathscr{V}}(x, y_1, \ldots, y_n, z_1, \ldots, z_n)|!$ works in a locally finite variety satisfying (‡).

In this section, we examine the following four properties and prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(i) Every subalgebra of $\mathbf{A}^2$ is quasi-Hamiltonian.
(ii) $\mathbf{A}$ satisfies (‡).
(iii) $\mathbf{A}$ satisfies (†).
(iv) $\mathbf{A}$ is quasi-Hamiltonian.

Our arguments depend on the following lemma, which gives a handy reformulation of the definition of "quasi-Hamiltonian".

LEMMA 3.2. $\mathbf{A}$ *is quasi-Hamiltonian if and only if whenever* C *is a subuniverse of* $\mathbf{A}$, $s \in C$ *and* $r \in A - C$ *is such that* $C \cup \{r\}$ *generates* $\mathbf{A}$, *then* $(r, s) \notin \mathrm{Cg}^{\mathbf{A}}(C \times C)$.

Proof. Assume that $\mathbf{A}$ is quasi-Hamiltonian, that C is a subuniverse of $\mathbf{A}$, that $s \in C$ and $r \in A - C$. By Zorn's Lemma, there is a subuniverse $D \subseteq A$ which contains C but does not contain r and is maximal for not containing r. Since $C \cup \{r\}$ generates $\mathbf{A}$, any such D is a maximal subuniverse of $\mathbf{A}$. $\mathbf{A}$ is quasi-Hamiltonian, so D is a congruence block of some congruence θ. We have $C \subseteq D$ and $r \notin D$, so we get that $C \times C \subseteq \theta$ and $(r, s) \notin \theta$. It follows that $(r, s) \notin \mathrm{Cg}^{\mathbf{A}}(C \times C)$.

Now assume that $\mathbf{A}$ is not quasi-Hamiltonian. Let C be any maximal subuniverse of $\mathbf{A}$ which is not a congruence block. Choose any $r \in A - C$ and any $s \in C$. Since C is a maximal, $C \cup \{r\}$ generates $\mathbf{A}$. Let $\theta = \mathrm{Cg}^{\mathbf{A}}(C \times C)$. The set C^{θ} is a subuniverse of $\mathbf{A}$ which contains C and is a θ-block. Since C is not a congruence block, but is a maximal subuniverse, we get that $C^{\theta} = A$. It follows that $\theta = 1_{\mathbf{A}}$ and so $(r, s) \in \theta = \mathrm{Cg}^{\mathbf{A}}(C \times C)$. □

LEMMA 3.3. *Let* $\mathbf{A}$ *be a finite algebra. If every subalgebra of* $\mathbf{A}^2$ *is quasi-Hamiltonian, then* $\mathbf{A}$ satisfies (‡).

Proof. We first establish the following claim concerning $\mathbf{A}$ under the assumption that subalgebras of $\mathbf{A}^2$ are quasi-Hamiltonian.

Claim. Choose $a_i, b_i, u, v \in A$ and $q(x, \bar{y}) \in \mathrm{Pol}_{n+1}\,\mathbf{A}$. If $q(u, \bar{b}) = u$ and $q(v, \bar{b}) = v$, then $(u, v) \in \mathrm{Cg}^{\mathbf{A}}(q(u, \bar{a}), q(v, \bar{a}))$.

Proof of Claim. Assume otherwise that $(u, v) \notin \alpha$ where $\alpha := \mathrm{Cg}^{\mathbf{A}}(q(u, \bar{a}), q(v, \bar{a}))$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^2$ generated by the set

$$\{(u, v)\} \cup \{(x, x) \mid x \in A\}.$$

The set $C = \{(x, y) \in B \mid (x, y) \in \alpha\}$ is a subuniverse of $\mathbf{B}$. We have $r := (u, v) \in B - C$ since $(u, v) \in \alpha$. $\mathbf{B}$ is generated by $C \cup \{r\}$. Set $s = (q(u, \bar{a}), q(v, \bar{a}))$ $(\in C)$. $\mathbf{B}$ has a polynomial $\hat{q}(x, \bar{y})$, equal to $q(x, \bar{y})$ acting coordinatewise, and with this polynomial we have

$$r = (u, v) = \hat{q}((u, v), \overline{(b_i, b_i)}) \operatorname{Cg}^{\mathbf{B}}(C \times C)\hat{q}((u, v), \overline{(a_i, a_i)}) = s,$$

since $\langle (a_i, a_i), (b_i, b_i) \rangle \in \operatorname{Cg}^{\mathbf{B}}(C \times C)$. But this contradicts the conclusion of Lemma 3.2: we should have $(r, s) \notin \operatorname{Cg}^{\mathbf{B}}(C \times C)$. This establishes the Claim.

Now assume that $\mathbf{A}$ has idempotent twins $e_{\bar{a}}^{\mathbf{A}}(x)$ and $e_{\bar{b}}^{\mathbf{A}}(x)$ such that $|e_{\bar{a}}^{\mathbf{A}}(A)| < |e_{\bar{b}}^{\mathbf{A}}(A)|$. Then $(e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{|A|!}(x)$ and $(e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{b}}^{\mathbf{A}})^{|A|!}(x) = e_{\bar{b}}^{\mathbf{A}}(x)$ are idempotent twins with

$$|(e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{|A|!}(A)| \leq |e_{\bar{a}}^{\mathbf{A}}(A)| < |e_{\bar{b}}^{\mathbf{A}}(A)|.$$

Furthermore, we now have $(e_{\bar{b}}^{\mathbf{A}} \circ e_{\bar{a}}^{\mathbf{A}})^{|A|!}(A) \subseteq e_{\bar{b}}^{\mathbf{A}}(A)$. Therefore, if there are idempotent twins with ranges of different sizes, then there are such twins where the range of one is properly contained in the range of the other. Changing notation back, we may assume that $e_{\bar{a}}^{\mathbf{A}}(A) \subset e_{\bar{b}}^{\mathbf{A}}(A)$. Choose $v \in e_{\bar{b}}^{\mathbf{A}}(A) - e_{\bar{a}}^{\mathbf{A}}(A)$ and set $u = e_{\bar{a}}^{\mathbf{A}}(v)$. We have $e^{\mathbf{A}}(u, \bar{b}) = u$, $e^{\mathbf{A}}(v, \bar{b}) = v$, so we can apply the Claim to deduce that

$$(u, v) \in \alpha := \operatorname{Cg}^{\mathbf{A}}(e^{\mathbf{A}}(u, \bar{a}), e^{\mathbf{A}}(v, \bar{a})).$$

But $e^{\mathbf{A}}(u, \bar{a}) = u = e^{\mathbf{A}}(v, \bar{a})$, which implies that $\alpha = 0_{\mathbf{A}}$ and so $u = v$. This is impossible, since $u \in e_{\bar{a}}^{\mathbf{A}}(A)$ and $v \notin e_{\bar{a}}^{\mathbf{A}}(A)$. This contradiction finishes the proof. □

LEMMA 3.4. *Assume that* $\mathbf{A}$ *satisfies* (‡). *Then* $\mathbf{A}$ *satisfies* (†).

Proof. Assume that (†) fails. This means that $\mathbf{A}$ has a prime quotient $\langle \delta, \theta \rangle$ such that $C(1, N^2; \delta)$ fails for some $\langle \delta, \theta \rangle$-trace N. From this we will construct a pair of idempotent twins of $\mathbf{A}$ which have ranges of different sizes. We argue by cases depending on $\operatorname{typ}(\delta, \theta)$.

Let U be a $\langle \delta, \theta \rangle$-minimal set containing N. Since $C(1, N^2; \delta)$ fails, we can find a polynomial $p \in \operatorname{Pol}_{n+1}\mathbf{A}$ and elements $a, b \in A$, $\bar{u}, \bar{v} \in N^n$ such that

$$p(a, \bar{u}) \; \delta \; p(a, \bar{v})$$

while

$$p(b, \bar{u}) \; \theta - \delta \; p(b, \bar{v}).$$

There is no loss of generality in assuming that $p(A, A^n) \subseteq U$.

Case 1. $\operatorname{typ}(\delta, \theta) = \mathbf{1}$.

Proof for Case 1. We claim that the polynomial $p(x, \bar{y})$ witnessing the fact that $C(1, N^2; \delta)$ fails can be chosen to be binary. To see that this is so, assume that $p(x, \bar{y})$ has minimal arity for a witness to $\neg C(1, N^2; \delta)$ and that arity is $n+1$ with $n>1$. Write $p_a(\bar{y})$ for the n-ary polynomial $p(a,\bar{y})$ and similarly write $p_b(\bar{y})$ for $p(b,\bar{y})$. Since $\mathrm{typ}(\delta,\theta)=\mathbf{1}$, θ is strongly abelian over δ. We also have $p_a(u_0,u_1,\ldots,u_{n-1})\;\delta\;p_a(v_0,v_1,\ldots,v_{n-1})$, so from the strong term condition we get that $p_a(u_0,v_1,\ldots,v_{n-1})\;\delta\;p_a(v_0,v_1,\ldots,v_{n-1})$. Further, $p_b(u_0,v_1,\ldots,v_{n-1})\;\theta\;p_b(v_0,v_1,\ldots,v_{n-1})$. If these last two elements are not δ-related, then we can choose $p'(x,y)=p(x,y,v_1,\ldots,v_n)$. Then

$$p'(a,u_0)\;\delta\;p'(a,v_0),$$

but

$$p'(b,u_0)\;\theta-\delta\;p'(b,v_0).$$

This p' is a binary witness to $\neg C(1,N^2;\delta)$. Therefore, assume that

$$p_b(u_0,v_1,\ldots,v_{n-1})\;\delta\;p_b(v_0,v_1,\ldots,v_{n-1}).$$

In this case,

$$p(a,u_0,u_1,\ldots,u_{n-1})\;\delta\;p(a,v_0,v_1,\ldots,v_{n-1})\;\delta\;p(a,u_0,v_1,\ldots,v_{n-1})$$

and

$$p(b,u_0,u_1,\ldots,u_{n-1})\;\theta-\delta\;p(b,v_0,v_1,\ldots,v_{n-1})\;\delta\;p(b,u_0,v_1,\ldots,v_{n-1}).$$

Let $p''(x,y_1,\ldots,y_{n-1})=p(x,u_0,y_1,\ldots,y_{n-1})$. The previous two displayed equations prove that p'' is a polynomial which witnesses the fact that $C(1,N^2;\delta)$ fails and p'' has arity smaller than the arity of p. Since this reduction can be accomplished whenever the arity of p is more than 2, we can assume that p is a binary polynomial.

We are at the point where we know that there is a polynomial $p(x,y)\in\mathrm{Pol}_2\mathbf{A}$ and elements $a,b\in A$, $u,v\in N$ such that

$$p(a,u)\;\delta\;p(a,v)$$

while

$$p(b,u)\;\theta-\delta\;p(b,v)$$

and $p(A,A)\subseteq U$. These equations imply that $p_a(\theta|_U)\subseteq\delta$ while $p_b(\theta|_U)\nsubseteq\delta$. The latter condition forces $p_b(y)$ to be a permutation of U. We can iterate $p(x,y)$ in its second variable until we obtain a binary polynomial $q(x,y)$ for which

$q(x, q(x, y)) = q(x, y)$ holds. We have $q_a(\theta|_U) \subseteq \delta$ while $q_b(y) = y$ on U. Since $q(x, y)$ is in the clone of operations generated by $p(x, \bar{y})$, and $p(A, A^n) \subseteq U$, we get that $q(A, A) \subseteq U$. Hence

- $q(x, q(x, y)) = q(x, y)$ and
- $q(a, A) \subset q(b, A) = U$.

The first item implies that $q(a, x)$ and $q(b, x)$ are idempotent twins of $\mathbf{A}$, while the second shows that they have ranges of different sizes. This is a failure of (‡), so the argument for Case 1 is complete.

Case 2. $\mathrm{typ}(\delta, \theta) = \mathbf{2}$.

Proof for Case 2. Choose an arbitrary pair $(u, v) \in N^2 - \delta$. Since $\mathbf{A}|_N$ is Mal'cev, and the congruence $1_{\mathbf{A}|_N}$ is generated by $\delta \cup (u, v)$ and contains the pairs (u_i, v_i), it is possible to find $r_i(x) \in \mathrm{Pol}_1 \mathbf{A}|_N$ such that $r_i(u)\ \delta\ u_i$ and $r_i(v)\ \delta\ v_i$ for all i. Define $p'(x, y) = p(x, r_1(y), \ldots, r_n(y))$. We have

$$p'(a, u)\ \delta\ p(a, \bar{u})\ \delta\ p(a, \bar{v})\ \delta\ p'(a, v)$$

while

$$p'(b, u)\ \delta\ p(b, \bar{u})\ \theta - \delta\ p(b, \bar{v})\ \delta\ p'(b, v).$$

Thus, $p'(a, \theta|_U) \subseteq \delta$ while $p'(b, \theta|_U) \nsubseteq \delta$. It follows that $p'(a, x)$ is not a permutation of U although $p'(b, x)$ is. Now arguments like those at the end of Case 1 show how to construct $q(x, y)$ such that

- $q(x, q(x, y)) = q(x, y)$ and
- $q(a, A) \subset q(b, A) = U$.

This is a failure of (‡), so we are done with Case 2.

Case 3. $\mathrm{typ}(\delta, \theta) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

Proof for Case 3. Choose $(0, 1) \in \theta|_U - \delta$, and a pseudo-meet polynomial $x \wedge y$ for U. On U, we have $1 \wedge x = x = x \wedge 1$, $x \wedge (x \wedge y) = x \wedge y$ and $x \wedge x = x$. For $a = 0$, $b = 1$ and $q(x, y) = x \wedge y$ we get (assuming $q(A, A) \subseteq U$, which we may) that

- $q(x, q(x, y)) = q(x, y)$ and
- $q(a, A) \subset q(b, A) = U$.

This finishes the argument for Case 3. The lemma is proved. □

LEMMA 3.5. *Let* $\mathbf{A}$ *be a finite algebra. If* $\mathbf{A}$ *satisfies* (†), *then* $\mathbf{A}$ *is quasi-Hamiltonian.*

Proof. Assume that there is some finite algebra $\mathbf{A}$ where $C(1, N^2; \delta)$ holds whenever $\delta \prec \theta$ in $\mathrm{Con}\, \mathbf{A}$ and N is a $\langle \delta, \theta \rangle$-trace, but $\mathbf{A}$ is not quasi-Hamiltonian. Choose such an $\mathbf{A}$ of minimum cardinality. Since condition (†) is inherited by homomorphic images, our minimality assumption implies that every proper homomorphic image of $\mathbf{A}$ is quasi-Hamiltonian. Let B be a maximal subuniverse of $\mathbf{A}$ which is not a congruence block and let γ be a minimal congruence. Neither our hypothesis nor our conclusion will be affected if we expand $\mathbf{A}$ by adding new constant operations to denote the elements of B, so we assume that every member of B is the interpretation of a constant term.

Case 1. B is a union of γ-blocks.

Proof for Case 1. In this case B/γ is a maximal (proper) subuniverse of $\mathbf{A}/\gamma$. By our minimality hypothesis, there is a congruence $\delta/\gamma \in \mathrm{Con}\,\mathbf{A}/\gamma$ which has B/γ as a congruence block. But this forces B to be a δ-block in $\mathbf{A}$; contrary to what we have assumed. Hence Case 1 canot occur.

Case 2. B is not a union of γ-blocks.

Proof for Case 2. $C(1, N^2; 0)$ holds where N is a $\langle 0, \gamma\rangle$-trace, so we have $C(N^2, N^2; 0)$. This implies that $\langle 0, \gamma\rangle$ is of type **1** or **2**. By Theorem 2.1, we cannot have $\gamma|_B > 0_{\mathbf{B}}$ if B is not a union of γ-blocks, so $\gamma|_B = 0_{\mathbf{B}}$. Since B is not a union of γ-blocks, $B^\gamma = A$. Hence for any $u \in A - B$ there is a $v \in B$ such that $(u, v) \in \gamma$. The γ-block containing $\{u, v\}$ is connected by traces, so we can find a $\langle 0, \gamma\rangle$-trace M which has an element $b \in M \cap B$ and an element $a \in M \cap (A - B)$. Fix such a choice of M, a and b.

$\mathrm{Cg}^{\mathbf{A}}(B \times B) = 1_{\mathbf{A}}$, so there must exist $c, d \in B$ and a unary polynomial $p(x)$ such that $p(c) \in B$ while $p(d) \in A - B$. We may express p as $p(x) = t^{\mathbf{A}}(x, a)$ for some term t. Now $t^{\mathbf{A}}(c, a) = p(c) \in B$ and $t^{\mathbf{A}}(c, a)$ is γ-related to $t^{\mathbf{A}}(c, b) \in B$. Since $\gamma|_B = 0_{\mathbf{B}}$, we conclude that

$$t^{\mathbf{A}}(c, a) = t^{\mathbf{A}}(c, b).$$

Since $C(1, M^2; 0)$ holds and $a, b \in M$, we get that

$$t^{\mathbf{A}}(d, a) = t^{\mathbf{A}}(d, b) \in B.$$

But this is false, since $t^{\mathbf{A}}(d, a) = p(d) \in A - B$. This contradiction finishes the proof of the theorem. □

Lemmas 3.3, 3.4 and 3.5 have the following consequence.

COROLLARY 3.6. *Let* $\mathbf{A}$ *be a finite algebra. If* $\mathbf{A}^2$ *is quasi-Hamiltonian, then* $\mathbf{A}$ *satisfies* (†) (*and* (‡)). *Conversely, if* $\mathbf{A}$ *satisfies* (†) (*or* (‡)), *then* $\mathbf{A}$ *is quasi-Hamiltonian.* □

As we have pointed out, for finite algebras the property (‡) is equivalent to an equational statement. Therefore, it is easy to say when a finite algebra generates a quasi-Hamiltonian variety. But first, we prove that the quasi-Hamiltonian property is local.

LEMMA 3.7. *A locally finite variety is quasi-Hamiltonian if and only if each finite member is.*

Proof. Let $\mathcal{V}$ be a locally finite variety whose finite members are quasi-Hamiltonian. For the purposes of obtaining a contradiction, assume that $\mathbf{B}$ is an algebra in $\mathcal{V}$ that is not quasi-Hamiltonian. According to Lemma 3.2, $\mathbf{B}$ has a subuniverse C and elements $s \in C$ and $r \in B - C$ where $C \cup \{r\}$ generates $\mathbf{B}$ and $(r, s) \in \mathrm{Cg}^{\mathbf{B}}(C \times C)$. From Mal'cev's congruence generation theorem, it is clear that $\mathbf{B}$ has a finitely generated subalgebra $\mathbf{B}'$ containing r and s such that, for $C' = B' \cap C$, it is the case that $(r, s) \in \mathrm{Cg}^{\mathbf{B}'}(C' \times C')$. Of course, we do not know whether $\mathbf{B}'$ is generated by $C' \cup \{r\}$. But since $C \cup \{r\}$ generates $\mathbf{B}$, there is a finite subset $U \subseteq C$ such that every member of the finite set B' is contained in the subuniverse of $\mathbf{B}$ generated by $U \cup \{r\}$. Let $\mathbf{B}''$ be the subalgebra of $\mathbf{B}$ generated by $U \cup \{r\}$ and let $C'' = B'' \cap C$. Since $C' \subseteq C''$ and $\mathbf{B}' \leq \mathbf{B}''$, we have $(r, s) \in \mathrm{Cg}^{\mathbf{B}''}(C'' \times C'')$. But now we have that $U \subseteq C''$ and $\mathbf{B}''$ is generated by $U \cup \{r\}$, so $\mathbf{B}''$ is generated by $C'' \cup \{r\}$. All conditions of Lemma 3.2 hold for the finitely generated algebra $\mathbf{B}''$ and they imply that this algebra is not quasi-Hamiltonian. This is impossible, since $\mathcal{V}$ is a locally finite variety whose finite members are quasi-Hamiltonian. □

COROLLARY 3.8. *For a finite algebra* $\mathbf{A}$, *the following conditions are equivalent*.

(i) *Every subalgebra of* $\mathbf{A}^2$ *satisfies* (†).
(ii) *Every subalgebra of* $\mathbf{A}^2$ *is quasi-Hamiltonian*.
(iii) $\mathbf{A}$ *satisfies* (‡).
(iv) *The variety generated by* $\mathbf{A}$ is quasi-Hamiltonian.
(v) *The finite members of the variety generated by* $\mathbf{A}$ *satisfy* (‡).
(vi) *The finite members of the variety generated by* $\mathbf{A}$ *satisfy* (†).

Proof. Lemma 3.5 proves that (i) $\Rightarrow$ (ii). Lemma 3.3 proves that (ii) $\Rightarrow$ (iii). For finite algebras, (iii) is equivalent to a statement about equations, so (iii) implies that every finite member of $\mathcal{V}(\mathbf{A})$ satisfies (‡), hence by Lemmas 3.4 and 3.5 we get that every finite member of $\mathcal{V}(\mathbf{A})$ is quasi-Hamiltonian. Lemma 3.7 then finishes the argument for (iii) $\Rightarrow$ (iv). Lemmas 3.3 and 3.4 prove that (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vi), respectively. The implication (vi) $\Rightarrow$ (i) is trivial. □

Corollary 3.8 shows how one can prove that a given finite algebra generates a quasi-Hamiltonian variety: one simply calculates whether or not all subalgebras of $\mathbf{A}^2$ satisfy (†). However, since the condition (†) is new, the following theorem, which describes a sufficient condition for $\mathscr{V}(\mathbf{A})$ to be quasi-Hamiltonian, may be more interesting.

THEOREM 3.9. *Let* **A** *be a finite algebra. If* **A** *is left nilpotent, then* $\mathscr{V}(\mathbf{A})$ *is quasi-Hamiltonian.*

Proof. By Corollary 3.8 (iii) ⇔ (iv), it will suffice to prove that idempotent twins of **A** have ranges of the same size. As we argued in Lemma 3.4, it is enough to show that there do not exist idempotent twins $e_{\bar{a}}^{\mathbf{A}}(x)$ and $e_{\bar{b}}^{\mathbf{A}}(x)$ such that $e_{\bar{a}}^{\mathbf{A}}(A) \subset e_{\bar{b}}^{\mathbf{A}}(A)$. Assume otherwise that such twins exist. Choose $v \in e_{\bar{b}}^{\mathbf{A}}(A) - e_{\bar{a}}^{\mathbf{A}}(A)$ and let $u = e_{\bar{a}}^{\mathbf{A}}(v)$. Define $\theta = \mathrm{Cg}^{\mathbf{A}}(u, v)$. Since

$$e^{\mathbf{A}}(u, \bar{a}) = u = e^{\mathbf{A}}(v, \bar{a}),$$

we get that

$$u = e^{\mathbf{A}}(u, \bar{b})\,[1, \theta]\, e^{\mathbf{A}}(v, \bar{b}) = v.$$

Therefore, $(u, v) \in [1, \theta] \leq \theta = \mathrm{Cg}^{\mathbf{A}}(u, v)$. We must have $[1, \theta] = \theta$. But in a left nilpotent algebra this implies that $\theta = 0_{\mathbf{A}}$. This forces $u = v$, which is impossible since $u \in e_{\bar{a}}^{\mathbf{A}}(A)$ and $v \notin e_{\bar{a}}^{\mathbf{A}}(A)$. This proof is finished. □

For a single finite algebra **A**, the condition (‡) (which requires idempotent twins to have ranges of the same size) is a necessary and sufficient condition for **A** to generate a quasi-Hamiltonian variety. Theorem 3.9 proves that the left nilpotence of **A** is a sufficient condition for $\mathscr{V}(\mathbf{A})$ to be quasi-Hamiltonian, but left nilpotence is not necessary. To see this, note that there are finite left nilpotent algebras which do not generate locally left nilpotent varieties. (Example 4 of [3] describes such an algebra.) Hence there exist finite algebras **A** such that **A** is not left nilpotent and yet $\mathscr{V}(\mathbf{A})$ is quasi-Hamiltonian; simply take **A** to be a finite non-nilpotent member of a variety generated by a finite left nilpotent algebra **B**. $\mathscr{V}(\mathbf{B})$ is quasi-Hamiltonian, so $\mathscr{V}(\mathbf{A})$ is too. Thus, it is not true that locally finite quasi-Hamiltonian varieties are locally left nilpotent, and so the sufficient condition of Theorem 3.9 is not necessary.

The condition (†) is a necessary condition for $\mathscr{V}(\mathbf{A})$ to be quasi-Hamiltonian, and (†) suffices to force the single algebra **A** to be quasi-Hamiltonian. However, having (†) hold in **A** is not sufficient for $\mathscr{V}(\mathbf{A})$ to be a quasi-Hamiltonian. Here are operation tables for a finite algebra **A** which satisfies (†), but does not satisfy (‡) (and therefore does not generate a quasi-Hamiltonian variety). This example has universe {0, 1, 2, 3, 4, 5}, a binary basic operation $*$ and two unary basic operations f and g.

$*$	0	1	2	3	4	5
0	0	1	2	1	0	5
1	0	1	2	1	0	5
2	0	1	2	1	0	5
3	0	1	2	1	0	5
4	0	1	2	1	0	5
5	0	1	2	3	4	5

	0	1	2	3	4	5
f	0	1	2	4	3	5
g	2	4	3	4	2	5

This algebra fails (‡) since $0 * x$ and $5 * x$ are idempotent twins with different size ranges. However, this algebra satisfies (†). There is a unique proper nontrivial congruence α which partitions the algebra as 01234/5. Furthermore, $\text{typ}(0, \alpha) = \text{typ}(\alpha, 1) = \mathbf{1}$. In particular, $C(1, 1; \alpha)$ holds, which implies that $C(1, N^2, \alpha)$ holds when N is a $\langle \alpha, 1 \rangle$-trace. The set $M = \{0, 1, 2\}$ is a $\langle 0, \alpha \rangle$-trace and it can be calculated that $C(1, M^2; 0)$ holds. It follows that **A** satisfies (†), although it fails (†).

REFERENCES

[1] HALL, M. JR., *The Theory of Groups*, Macmillan, 1959.

[2] HOBBY, D. and MCKENZIE, R., *The Structure of Finite Algebras*, Contemporary Mathematics vol. 76, American Mathematical Society, 1988.

[3] KEARNES, K. A., *An order-theoretic property of the commutator*, International Journal of Algebra and Computation *3* (1993), 1–43.

[4] KEARNES, K. A., *Locally solvable factors of varieties*, Proc. Amer. Math. Soc. 124 (1996), 3619–3625.

[5] KEARNES, K. A., KISS, E. W. and VALERIOTE, M. A., *Minimal sets and varieties*, to appear in Trans. Amer. Math. Soc.

[6] KEARNES, K. A. and SZENDREI, Á., *A characterization of minimal locally finite varieties*, to appear in Trans. Amer. Math. Soc.

[7] KISS, E. W. and VALERIOTE, M., *Abelian algebras and the Hamiltonian property*, Journal of Pure and Applied Algebra *87* (1993), 37–49.

[8] KLUKOVITS, L., *Hamiltonian varieties of universal algebras*, Acta Sci. Math. *37* (1975), 11–15.

[9] MCKENZIE, R., *Congruence extension, Hamiltonian and Abelian properties in locally finite varieties*, Algebra univers. *28* (1991), 589–603.

[10] MCKENZIE, R., MCNULTY, G. and TAYLOR, W., *Algebras, Lattices, Varieties Volume 1*, Wadsworth and Brooks/Cole, Monterey, California, 1987.

[11] MCKENZIE, R. and VALERIOTE, M., *The Structure of Decidable Locally Finite Varieties*, Progress in Mathematics v. 79, Birkhäuser, 1989.

[12] SZENDREI, Á., *Maximal non-affine reducts of simple affine algebras*, Algebra univers. *34* (1995), 144–174.
[13] SZENDREI, Á., *Strongly abelian minimal varieties*, Acta Sci. Math. *59* (1994), 25–42.
[14] VALERIOTE, M., *Finite simple abelian algebras are strictly simple*, Proc. Amer. Math. Soc. *108* (1990), 49–57.

1498 N. Timberline Dr.
Fayetteville, AR 72704
U.S.A.

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