Natural Examples of Quasivarieties With EDPM

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A quasivariety $\mathcal{K}$ has **equationally definable principal meets**, or **EDPM**, if there are finitely many pairs of terms $(p_i(x,y,z,u), q_i(x,y,z,u))$, $i < n$, such that for any $\mathbf{A} \in \mathcal{K}$ and $a,b,c,d \in \mathbf{A}$ we have

$$\theta_{\mathcal{K}}(a,b) \cdot \theta_{\mathcal{K}}(c,d) \;=\; \theta_{\mathcal{K}}(\{(p_i(a,b,c,d), q_i(a,b,c,d)) | i \in I\}).$$

Here $\theta_{\mathcal{K}}(X)$ denotes the least congruence θ containing X such that $\mathbf{A}/\theta \in \mathcal{K}$. Quasivarieties with EDPM have recently arisen in the study of finitely based quasivarieties. They are known to be **relatively congruence distributive**, which means that the lattice of congruences on $\mathbf{A}$ of the form $\theta_{\mathcal{K}}(X)$ is a distributive lattice for any $\mathbf{A} \in \mathcal{K}$.

Few examples of quasivarieties with EDPM which did not lie in a congruence distributive variety (or at least a modular variety) were known until it was proved in [2] that any finite **order-primal** algebra generates a relatively distributive quasivariety. (An algebra $\mathbf{A}$ is order-primal if there exists a partial ordering, $\langle A, \leq \rangle$, of the universe of $\mathbf{A}$ such that the terms of $\mathbf{A}$ are precisely the operations on A which are monotone with respect to $\leq$.) Since an order-primal algebra $\mathbf{A}$ has no non-trivial subalgebras, the only finitely subdirectly irreducible algebra in $\mathbf{SP}(\mathbf{A})$ up to isomorphism is $\mathbf{A}$. The class of finitely subdirectly irreducible algebras forms a universal class, so it follows from Theorem 2.3 $(i) \leftrightarrow (iv)$ of [1] that the quasivariety generated by a finite order-primal algebra has EDPM. The result in [2], that finite order-primal algebras generate relatively distributive quasivarieties, uses a fairly long argument involving generalized duality theory. The same result was later proved in [3] using tame congruence theory. We now give a short, direct proof that a certain class of finite algebras (including all the order-primal algebras) generate quasivarieties with EDPM.

Theorem *If* $\mathbf{A}$ *is a finite algebra and* $\langle A, \leq \rangle$ *is a partial order such that every* 4*-ary operation on* A *which is monotone with respect to* $\leq$ *is a term of* $\mathbf{A}$*, then* $\mathcal{K} = \mathbf{SP}(\mathbf{A})$ *is a quasivariety with* EDPM*. Hence,* $\mathcal{K}$ *is a relatively distributive quasivariety.*

Proof: If $\mathbf{A}$ satisfies the hypotheses of the Theorem, then $\mathbf{A}$ is subdirectly irreducible and has no non-trivial subalgebras. It is easy to prove (and this result is Theorem 2.3 $(i) \leftrightarrow (vii)$ of [1]) that the pairs (p_i, q_i), $i < n$, are terms witnessing EDPM for $\mathcal{K}$ iff

$$\mathbf{A} \models \forall x,y,z,u(\bigwedge_{i<n}(p_i(x,y,z,u) = q_i(x,y,z,u)) \Leftrightarrow x = y \;\text{ or }\; z = u).$$

If we let $I = \{(a,b,c,d) \in A^4 | \, a \neq b \;\text{ and }\; c \neq d\}$ we may rewrite this as

$$\mathbf{A} \models \forall \bar{x}(\bigwedge_{i<n}(p_i(\bar{x}) = q_i(\bar{x})) \Leftrightarrow \bar{x} \notin I).$$

We will construct such pairs (p_i, q_i).

We may assume that $\mathbf{A}$ is non-trivial. If no two elements of $\mathbf{A}$ are $\leq$-comparable, then every binary operation on A is a term of $\mathbf{A}$. A classical result of Sierpinski implies that every finitary operation on A is a term of $\mathbf{A}$. We can change the ordering on A so that two elements are comparable and still retain the hypotheses of this theorem. Hence we may assume that $u \neq v$ are elements of A such that $u \leq v$. Recall the definition of I. For each $i \in I$ we define functions $p_i, q_i : A^4 \to A$ by:

$$p_i(\bar{x}) = \begin{cases} u & \text{if } \bar{x} \leq i \\ v & \text{otherwise} \end{cases} \quad \text{and } q_i(\bar{x}) = \begin{cases} p_i(\bar{x}) & \text{if } \bar{x} \neq i \\ v & \text{if } \bar{x} = i \end{cases}$$

The order on A^4 in this definition is the product order. p_i and q_i are monotone, therefore equal to terms, and $p_i(\bar{x}) = q_i(\bar{x})$ iff $\bar{x} \neq i$. Hence,

$$\mathbf{A} \models \forall \bar{x} (\bigwedge_{i \in I} (p_i(\bar{x}) = q_i(\bar{x})) \Leftrightarrow \bar{x} \notin I).$$

Since $|I|$ is finite, we have shown that $\mathcal{K}$ has EDPM. By Theorem 2.3 of [1], the fact that $\mathcal{K}$ is relatively congruence distributive follows from the fact that it has EDPM. □

Corollary *Every finite order-primal algebra generates a quasivariety with EDPM.* □

It is known that not every order-primal algebra lies inside a modular variety. The order primal algebras that lie inside a modular variety are characterized in [3].

References

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[3] R. McKenzie, *Monotone clones, residual smallness and congruence distributivity,* Bull. Austral. Math. Soc. **41** (1990), 283–300.

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