Subdirectly Irreducible Modes *†

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Abstract

We prove that subdirectly irreducible modes come in three very different types. From the description of the three types we derive the results that a variety of modes has a semilattice term if and only if it contains no nontrivial abelian algebras, and that a variety of modes has a Mal'cev term if and only if it contains no algebra which term equivalent to a 2-element set or 2-element semilattice.

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1 Introduction

A **mode** is an algebra which satisfies the idempotent and entropic laws. The **idempotent laws** for $\mathbf{M}$ assert that if f is any term of $\mathbf{M}$, then $\mathbf{M} \models f(x, \ldots, x) = x$. The **entropic laws** assert that if f is an m-ary term operation of $\mathbf{M}$ and g is an n-ary term operation of $\mathbf{M}$, then f and g commute on any $m \times n$ array of elements of $\mathbf{M}$. Together the idempotent and entropic laws are equivalent to the property that any polynomial operation $p = p(x_1, \ldots, x_n) \in \mathrm{Pol}(\mathbf{M})$ is a multivariable endomorphism; i.e., $p : \mathbf{M}^n \to \mathbf{M}$ is a homomorphism.

Let $\mathbf{S}$ be a subdirectly irreducible mode with monolith μ. If M is a nontrivial μ-class, then we will see that the subalgebra $\mathbf{M}$ supported by M is term equivalent to a set, a 2-element semilattice, or a quasi-affine algebra which is not strongly abelian. Our main result concerning the three types is that if $\mathbf{M}$ is a 2-element semilattice, then $\mathbf{S}$ itself has a semilattice term. A secondary result is that if $\mathbf{M}$ is a quasi-affine algebra which is not strongly abelian, then $\mathbf{S}$ has a nontrivial center.

We apply the results on subdirectly irreducible algebras to derive results on varieties of modes. We prove that a variety of modes has a semilattice term if and only if it contains no abelian algebras, and that it has a Mal'cev term if and only if it contains no algebra term equivalent to a 2-element set or a 2-element semilattice.

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2 Three Types

Throughout this paper **S** will denote a subdirectly irreducible mode with monolith μ, M denotes an arbitrarily chosen nontrivial μ-class, and **M** denotes the subalgebra supported by M. We will classify subdirectly irreducible modes according to the properties of the clone of **M**. (Throughout, when we refer to "the clone" of an algebra or variety, we always mean the clone of term operations.)

LEMMA 2.1 ([10], Proposition 1.12) *Let* **A** *be any algebra. The clone of* **A** *has at least one of the following properties.*

(0) Every operation is a projection.

(1) There is a unary operation which is not a projection.

(2) There is an idempotent essentially binary operation.

(3) There is an operation $M(x, y, z)$ satisfying the majority laws:

$$M(x, x, y) = M(x, y, x) = M(y, x, x) = x.$$

(4) There is an operation $m(x, y, z)$ satisfying the minority laws:

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = y.$$

(5) There is an operation $s(x_1, \ldots, x_n)$ of arity $n \geq 3$, which depends on all variables, and which satisfies the semiprojection laws: for any $1 \leq i < j \leq n$

$$s(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) = x_1.$$

COROLLARY 2.2 *The clone of* **M** *has exactly one of the following properties.*

(0) It is a clone of projections.

(2) It has an idempotent essentially binary operation.

(4)' It is the clone of an affine Boolean group. (I.e., it is generated by $x + y + z \pmod 2$.)

Proof: It is trivial to check that no clone has more than one of these properties, so we show only that one of these cases must occur.

Since **M** is idempotent, it has no unary operation different from a projection. Therefore case (1) of Lemma 2.1 cannot occur.

A majority operation does not commute with itself on any array of the following form:

$$\begin{bmatrix} x & x & x \\ x & y & y \\ y & y & x \end{bmatrix}, \qquad x \neq y.$$

Therefore case (3) of Lemma 2.1 cannot occur.

Assume that the clone of $\mathbf{M}$ contains a semiprojection, $s(x_1, \ldots, x_n)$. Since this operation depends on all of its variables, we can choose $a_i \in M$ such that $s(a_1, a_2, \ldots, a_n) \neq a_1$. The unary polynomial $\varepsilon(x) = s(a_1, x, a_3, \ldots, a_n)$ is an endomorphism of of $\mathbf{S}$. The semiprojection equations imply that the a_i are distinct from one another, and also that $(a_1, a_2) \notin \ker(\varepsilon)$ while $(a_1, a_3) \in \ker(\varepsilon)$. Since

$$(a_1, a_2), (a_1, a_3) \in M^2 \subseteq \mu,$$

this implies that $\ker(\varepsilon)$ is incomparable with μ, contradicting the subdirect irreducibility of $\mathbf{S}$. This shows that case (5) of Lemma 2.1 cannot occur.

We have shown so far that the clone of $\mathbf{M}$ is described by one of the cases (0), (2) or (4) of Lemma 2.1. Now we need to refine (4) to (4)'. Assume that $\mathbf{M}$ has a minority operation $m(x, y, z)$. Then $m(x, y, z)$ is a Mal'cev operation which commutes with all other operations in the clone of $\mathbf{M}$. It follows that $\mathbf{M}$ is an affine algebra. If $\mathbf{M}$ is affine over the ring $\mathbf{R}$, then every operation of the form $rx + (1 - r)y$ is in the clone of $\mathbf{M}$. Thus, either the clone of $\mathbf{M}$ has an idempotent essentially binary operation (which is case (2)) or else $\mathbf{R}$ is the 2-element ring. In the latter case, $\mathbf{M}$ is an idempotent algebra which is affine over the 2-element ring, so its clone is generated by $x + y + z \pmod 2$. $\square$

We are going to classify subdirectly irreducible modes into three types, called the "set type", the "semilattice type" and the "quasi-affine type". We say that $\mathbf{S}$ is of **set type** if the clone of $\mathbf{M}$ is a clone of projections. $\mathbf{S}$ is of **semilattice type** if the clone of $\mathbf{M}$ has a noncancellative, essentially binary operation. $\mathbf{S}$ is of **quasi-affine type** if the clone of $\mathbf{M}$ has a cancellative binary operation, or else the clone is generated by $x + y + z \pmod 2$. The reasons behind the terminology will become clear as we go along.

The definitions just given do not depend on the choice of M for the following reasons. Choose any other nontrivial μ-class M'. If $a \neq b$ are in M and $a' \neq b'$ are in M', then the congruences $\mathrm{Cg}(a, b)$ and $\mathrm{Cg}(a', b')$ are equal to μ, so there are unary polynomials $p, p' \in \mathrm{Pol}(\mathbf{S})$ such that $p(a) \neq p(b) \in M'$ and $p'(a') \neq p'(b') \in M$. These polynomials are endomorphisms of $\mathbf{S}$ which are nonconstant on μ. Since $\mu \not\leq \ker(p), \ker(p')$, it follows that p and p' are injective on $\mathbf{S}$. Hence $p : \mathbf{M} \to \mathbf{M}'$ and $p' : \mathbf{M}' \to \mathbf{M}$ are embeddings, so the algebras $\mathbf{M}$ and $\mathbf{M}'$ embed into one another. This is enough to show that the "set type" is well defined. If $b(x, y)$ is a noncancellative, essentially binary operation of $\mathbf{M}$, then it is noncancellative and essentially binary on $\mathbf{M}'$ because $\mathbf{M}$ is embeddable in $\mathbf{M}'$. This is enough to show that the "semilattice type" is well defined. If $b(x, y)$ is a cancellative binary operation of $\mathbf{M}$, then it is cancellative on $\mathbf{M}'$ because $\mathbf{M}'$ is embeddable in $\mathbf{M}$. If the clone of $\mathbf{M}$ is generated by $x + y + z \pmod 2$ if and only if the same is true for $\mathbf{M}'$. This shows that the "quasi-affine type" is well defined.

If $\mathbf{S}$ is a finite subdirectly irreducible mode, then it is easy to show that $\mathbf{S}$ is a "$\langle 0, \mu \rangle$-minimal algebra", in the sense defined in [1]. (In the finite case, our three "types" of subdirectly irreducibles correspond to $\mathrm{typ}(0, \mu) = \mathbf{1}, \mathbf{2}$ or $\mathbf{5}$.) Tame congruence theory applied to finite modes leads quickly to nice structural results, which appear in [6]. We do not repeat those results here. Instead it is our goal to determine how much of what is true for the finite case extends to the infinite case.

Our main goal in this section is to prove that $\mathbf{S}$ has the semilattice type if and only if $\mathbf{S}$ has a semilattice term. Subdirectly irreducible modes with a semilattice term are understood fairly well. (See [4] and [5].) As a secondary goal we prove some results about **S** when it has quasi-affine type. Here our main results are that **S** has quasi-affine type if and only if **M** is a quasi-affine algebra which is not term equivalent to a set. Moreover, when this happens, then the center of **S** is a nonzero congruence which coincides with the centralizer of the monolith. Unlike the case when **S** has semilattice type, the results we prove here for arbitrary subdirectly irreducible modes of quasi-affine type fall far short of what is known for finite subdirectly irreducible modes of quasi-affine type. Furthermore, we prove nothing about subdirectly irreducible modes of set type, but little is known even for finite subdirectly irreducible modes of set type. (Some information can be found in [6] and [9].) Nevertheless, the results we obtain are strong enough for some interesting applications, which one can find in Section 3 and in [8].

THEOREM 2.3 **S** *has semilattice type if and only if it has a semilattice term*

Proof: If **S** has a semilattice term, then so does **M**, and this term is an example of a noncancellative, idempotent, essentially binary operation in the clone of **M**. Conversely, assume that $x \wedge y$ is a term of **S** which is noncancellative and essentially binary on M. We will prove that the term $x \wedge y$ interprets as a semilattice operation on **S**.

Claim 2.4 $|M| = 2$ and $x \wedge y$ is a semilattice operation on M.

For $u \in M$, define $L_u(x) = u \wedge x$ and $R_u(x) = x \wedge u$. Since $x \wedge y$ is noncancellative on M, there is a $0 \in M$ such that either $L_0(x)$ is not injective or $R_0(x)$ is not injective. Assuming the former, we get that $\mu \leq \ker(L_0)$, because L_0 is a noninjective endomorphism. Thus $L_0(x)$ is constant on M, yielding that $0 \wedge x = 0 \wedge 0 = 0$ if $x \in M$. This shows that 0 is a left zero element in M with respect to $\wedge$. Now select any $u \neq v \in M$. Since $M^2 \subseteq \mu = \mathrm{Cg}(u, v)$, it follows that there is a unary polynomial $p \in \mathrm{Pol}(\mathbf{S})$ such that $0 = p(u) \neq p(v)$ or $0 = p(v) \neq p(u)$. In either case, $(u, v) \notin \ker(p)$, so $\mu \not\leq \ker(p)$, and this implies that p is injective. Assuming that we are in the case $0 = p(u) \neq p(v)$, we get that

$$\begin{aligned} p(u \wedge v) &= p(u) \wedge p(v) \\ &= 0 \wedge p(v) \\ &= 0 \wedge p(u) \\ &= p(u) \wedge p(u) \\ &= p(u \wedge u) = p(u). \end{aligned}$$

Since p is injective, we must have $u \wedge v = u = u \wedge u$, so $(u, v) \in \ker(L_u)$. Thus L_u is not injective on M, and as in our earlier arguments for the element 0 this means that u is a left zero element with respect to $\wedge$. If, on the other hand, we are in the case $0 = p(v) \neq p(u)$, then we obtain that v is a left zero element for $\wedge$. This shows that whenever $u \neq v \in M$, then one of these two elements is a left zero element for $\wedge$.

If all elements of M were left zero elements with respect to $\wedge$, then $x \wedge y = x$ would be an equation holding in **M**. This would contradict the fact that $\wedge$ is essentially binary on M. Therefore, there is a unique element $1 \in M$ which is not a left zero element. Equivalently, 1 is the unique element $u \in M$ for which L_u is injective.

Assume that there is an element $u \in M - \{0, 1\}$. Since $(0, 1) \in \mu = \mathrm{Cg}(0, u)$, there is a unary polynomial p such that $1 = p(0) \neq p(u) =: r$ or $1 = p(u) \neq p(0) =: r$. The argument is the same in both cases, since both 0 and u are left zeros, so assuming that we are in the case $1 = p(0) \neq p(u) =: r$ we have

$$\begin{aligned} L_1(1) &= 1 \wedge 1 \\ &= 1 \\ &= p(0) \\ &= p(0 \wedge u) \\ &= p(0) \wedge p(u) \\ &= L_1(r). \end{aligned}$$

This forces L_1 to be noninjective, contrary to the choice of 1. Thus $M = \{0, 1\}$ and $\wedge$ is an idempotent operation on M for which 0 is a left zero and 1 is not a left zero. Necessarily $0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0$ and $1 \wedge 1 = 1$. Thus, $\wedge$ is a meet semilattice operation on M with respect to the ordering $0 < 1$. (This implies, in particular, that R_1 is injective on M, and hence on S.) This ends the proof of Claim 2.4.

Claim 2.5 M is the only nontrivial μ-class.

Choose a nontrivial μ-class M' independently of the choice of M. Then, since $\mathbf{M}$ and $\mathbf{M}'$ mutually embed into one another, therefore $|M'| = 2$ and $\wedge$ is a semilattice operation on M'. Denote the elements of M' by $0'$ and $1'$ so that $\wedge$ is a meet operation for the order $0' < 1'$. We now argue that $0 = 0'$. Since M' was chosen independently of M, this will show that M is the only nontrivial μ-class.

Subclaim 2.6 For any unary polynomial $p \in \mathrm{Pol}(\mathbf{S})$, if $u = p(0)$ and $v = p(0')$, then $L_u(x)$ is injective if and only if $L_v(x)$ is injective.

Assume that $L_u(x)$ is injective. Then $L_u(0) \neq L_u(1)$, and since L_u is a polynomial of $\mathbf{S}$ we get that $M'' := \{L_u(0), L_u(1)\}$ is a nontrivial μ-class. Since L_u is an endomorphism we get that $0'' := L_u(0) < L_u(1) =: 1''$ in the $\wedge$-order, considering $\wedge$ to be a meet operation on M''. Thus $p(0) \wedge 0 = 0'' < 1'' = p(0) \wedge 1$. We also have that $p(1) \wedge 1 \in M''$, since $p(1) \wedge 1 \equiv_\mu p(0) \wedge 1 = 1'' \in M''$. The endomorphism $p(x) \wedge 1$ preserves $\wedge$, so $1'' = p(0) \wedge 1 \leq p(1) \wedge 1$ in M''. This shows that

$$p(0) \wedge 0 < p(0) \wedge 1 = p(1) \wedge 1.$$

In particular, $R_1(p(0)) = R_1(p(1))$. We must have $p(0) = p(1)$, since $R_1(x)$ is injective, so $\mu \leq \ker(p)$. Moreover, by the last displayed line the endomorphism $h(x) := p(x) \wedge x$ satisfies $h(0) < h(1)$. Since $\mu \not\leq \ker(h)$ and $\mu \leq \ker(p)$ we must have

$$p(0') \wedge 0' = h(0') < h(1') = p(1') \wedge 1' = p(0') \wedge 1'.$$

Thus, $(0', 1') \notin \ker(L_v)$, which implies that L_v is injective. This finishes the proof of the subclaim.

Now we continue the proof of Claim 2.5. Choose $(r, s) \in \mu \leq \mathrm{Cg}(0, 0')$. There is a sequence $r = a_0, a_1, \ldots, a_n = s$ of distinct elements of S and a sequence $p_1, \ldots, p_n$ of unary polynomials of $\mathbf{S}$ such that $\{p_i(0), p_i(0')\} = \{a_{i-1}, a_i\}$. By the subclaim we know that, for each i, $L_{a_{i-1}}$ is injective if and only if L_{a_i} is injective. Therefore L_r is injective if and only if L_s is injective. Since L_1 is injective while L_0 is not, we conclude that $(0,1) \notin \mathrm{Cg}(0,0')$. But since $\mathrm{Cg}(0,1) = \mu$, this forces $0 = 0'$. Claim 2.5 is proved.

Claim 2.7 If $u \in S - \{1\}$, then $R_u(0) = R_u(1)$.

If $R_u(0) \neq R_u(1)$, then $\{R_u(0), R_u(1)\}$ is a nontrivial μ-class, which must be M according to Claim 2.5. Also, since R_u is an endomorphism, $R_u(0) < R_u(1)$ in the $\wedge$-order. Hence $R_u(1) = 1 \wedge u = 1$. But now $L_1(u) = 1 \wedge u = 1 = L_1(1)$. Since L_1 is injective we conclude that $u = 1$.

Claim 2.8 If $u, v \in S$ and $u \wedge v = 1$, then $u = v = 1$.

We know that $L_1(x)$ is injective and $L_1(1) = 1 \wedge 1 = 1$, therefore $L_1(v) = 1 \wedge v = 1 = L_1(1)$ implies $v = 1$. Similarly, $R_1(u) = u \wedge 1 = 1 = R_1(1)$ implies $u = 1$. Therefore we only need to prove that $u \wedge v = 1$ is impossible if $u \neq 1 \neq v$. Assume otherwise. By Claim 2.7, the assumption that $u \neq 1 \neq v$ implies that $1 \wedge u = R_u(1) = R_u(0) = 0 \wedge u$, and similarly $1 \wedge v = R_v(1) = R_v(0) = 0 \wedge v$. Thus if $u \wedge v = 1$ and $u \neq 1 \neq v$ we get

$$\begin{aligned} 1 &= 1 \wedge 1 \\ &= (1 \wedge 1) \wedge (u \wedge v) \\ &= (1 \wedge u) \wedge (1 \wedge v) \\ &= (0 \wedge u) \wedge (0 \wedge v) \\ &= (0 \wedge 0) \wedge (u \wedge v) \\ &= 0 \wedge 1 = 0, \end{aligned}$$

which is false. This proves Claim 2.8.

Claim 2.9 The term $x \wedge y$ interprets as a semilattice operation on $\mathbf{S}$.

We know $\mathbf{S} \models x \wedge x = x$, so we only need to verify that $\wedge$ is commutative and associative. To prove that $\mathbf{S} \models x \wedge y = y \wedge x$, assume that for some $u, v \in S$ we have $u \wedge v \neq v \wedge u$. Since $(0,1) \in \mathrm{Cg}(u \wedge v, v \wedge u)$ there must be a unary polynomial $p \in \mathrm{Pol}(\mathbf{S})$ such that $1 = p(u \wedge v) \neq p(v \wedge u)$, or the same with u and v interchanged. The situations are identical, so assume that $1 = p(u \wedge v) = p(u) \wedge p(v)$. By Claim 2.8 we conclude that $p(u) = p(v) = 1$. But now $1 \neq p(v \wedge u) = p(v) \wedge p(u) = 1 \wedge 1$, which is false. Thus $\mathbf{S} \models x \wedge y = y \wedge x$. The associative law is proved with the same kind of argument. □

Now we turn our attention to subdirectly irreducible modes of quasi-affine type. Recall that in the definition of this type there are two cases: $\mathbf{M}$ has a cancellative binary term or the clone of $\mathbf{M}$ is generated by $x+y+z \pmod 2$. In order to treat these cases with the same arguments, choose and fix $0 \in M$. If we are in the case that the clone of $\mathbf{M}$ is generated by $x + y + z \pmod 2$, then define $x * y = x + y + 0 \pmod 2$. In the other case, let $x * y$ be any cancellative binary term of $\mathbf{M}$. In either case, $x * y$ is a polynomial operation of $\mathbf{S}$, and of $\mathbf{M}$, which has the following properties.

- $x * y$ commutes with itself and with all term operations of $\mathbf{S}$.

- $x * y$ is cancellative on $\mathbf{M}$.
- either $\mathbf{M} \models x * x = x$ or $\mathbf{M} \models x * y = y * x$.
- $0 * 0 = 0$.

LEMMA 2.10 *Assume that* $\mathbf{S}$ *is of quasi-affine type. Then the monolith* μ *of* $\mathbf{S}$ *is an abelian congruence which is not strongly abelian.*

Proof: We refer the reader to Chapter 3 of [1] for the definition and properties of abelian and strongly abelian congruences.

Consider all matrices of the form

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad = \quad \begin{bmatrix} t(a, \mathbf{u}) & t(a, \mathbf{v}) \\ t(b, \mathbf{u}) & t(b, \mathbf{v}) \end{bmatrix}$$

where t is a term and $(a, b), (u_i, v_i) \in \mu$. If there is any such matrix with $r \neq s$, then there is a unary polynomial $f \in \mathrm{Pol}(\mathbf{S})$ such that

$$\begin{bmatrix} p' & q' \\ r' & s' \end{bmatrix} \quad = \quad \begin{bmatrix} fp & fq \\ fr & fs \end{bmatrix},$$

$r' \neq s'$ and $p' \equiv_\mu q' \equiv_\mu r' \equiv_\mu s' = 0 \in M$. Thus, if there is such a matrix with $p = q$ and $r \neq s$, then there is one with all entries in M.

Assume that there is such a matrix with $p = q$ and $r \neq s$ and all entries in M. Then since $x * y$ commutes with t we have

$$\begin{aligned} (p * q) * (r * s) &= t((a * a) * (b * b), (\mathbf{u} * \mathbf{v}) * (\mathbf{u} * \mathbf{v})) \\ &= t((a * a) * (b * b), (\mathbf{u} * \mathbf{u}) * (\mathbf{v} * \mathbf{v})) \\ &= (p * p) * (s * s). \end{aligned}$$

But $p = q$, so $(p * p) * (r * s) = (p * p) * (s * s)$. Since $p * p, r * s, s * s \in M$ and $x * y$ is left cancellative on M we get $r * s = s * s$. Using right cancellativity we get $r = s$, a contradiction. Thus there is no matrix with $p = q$ and $r \neq s$. This is exactly what it means to be abelian.

To see that μ is not strongly abelian, it suffices to show that $\mathbf{M}$ is not a strongly abelian algebra. Any binary polynomial $x*y$ of a strongly abelian algebra which satisfies $x*y = y*x$ must be constant. Any binary polynomial of a strongly abelian algebra which satisfies $x*x = x$ also satisfies $(x*y)*(u*v) = x*v$. $\mathbf{M}$ has a cancellative polynomial satisfying one of these conditions. Both possibilities lead to the conclusion that $\mathbf{M}$ has only one element, which is false. Thus μ is not strongly abelian. □

We retain all our previous notation: $\mathbf{S}, \mu, \mathbf{M}, 0, *$, etc, and add the following notation and terminology. If α is a congruence on an algebra, then the **centralizer of** α, denoted $(0 : \alpha)$, is the largest congruence β such that $[\beta, \alpha] = 0$. The centralizer of the total congruence is called the **center**, and is denoted ζ. Let $Q \subseteq S$ be the set of all elements $u \in S$ for which the mappings $L_u : S \to S : x \mapsto u * x$ and $R_u : S \to S : x \mapsto x * u$ are injective. Since $x * y$ is cancellative on M, it follows that $M \subseteq Q$. We say that a congruence θ of $\mathbf{S}$ is **contained** in a subset $X \subseteq S$ provided that X is a union of θ-classes.

LEMMA 2.11 *Let* $\mathbf{S}$ *be a subdirectly irreducible mode of quasi-affine type. Then* $(0:\mu) = \zeta$*, and this congruence is the largest congruence contained in* Q*.*

Proof: Choose $(u, v) \in (0:\mu)$ with $u \in Q$. Since $(u, v) \in (0:\mu)$,

$$M^2 \subseteq \ker(L_u) \Leftrightarrow M^2 \subseteq \ker(L_v) \quad \& \quad M^2 \subseteq \ker(R_u) \Leftrightarrow M^2 \subseteq \ker(R_v).$$

But for any unary polynomial p of $\mathbf{S}$, p is injective if and only if $M^2 \not\subseteq \ker(p)$. Since $u \in Q$ we have that L_u and R_u are injective. Consequently L_v and R_v are injective, too, proving that $v \in Q$. Therefore $(0:\mu)$ is contained in Q. If λ denotes the largest congruence contained in Q, then this proves that $(0:\mu) \leq \lambda$. In addition to this, we have $\zeta := (0:1) \leq (0:\mu)$ simply because $\mu \leq 1$. Thus, $\zeta \leq (0:\mu) \leq \lambda$. Now we show that $\lambda \leq \zeta$.

Assume instead that $\lambda \not\leq \zeta$, so there is some $(a, b) \in \lambda - \zeta$. Since $(a, b) \notin \zeta$, there is a matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad = \quad \begin{bmatrix} t(a, \mathbf{u}) & t(a, \mathbf{v}) \\ t(b, \mathbf{u}) & t(b, \mathbf{v}) \end{bmatrix}$$

where t is a term, $u_i, v_i \in S$, and $p = q$ while $r \neq s$ (or the same condition with a and b switched). Since $\mathbf{S}$ is subdirectly irreducible with $0 \in M$, a μ-class, we may assume that $s = 0$ (as we argued in the proof of Lemma 2.10). But now $r \equiv_\lambda p = q \equiv_\lambda s = 0 \in M \subseteq Q$. Since λ is contained in Q and $0 \in Q$, we get that the congruence class $0/\lambda$, which contains p, q, r and s, is a subset of Q. Moreover, since $0 * 0 = 0$, the congruence class $0/\lambda$ is closed under $*$. Therefore $p * p, r * s, s * s \in 0/\lambda \subseteq Q$. Now we can copy the argument of Lemma 2.10: If we have any matrix of the above form where $p = q$ and $r \neq s$, then we get $(p * p) * (r * s) = (p * q) * (r * s) = (p * p) * (s * s)$. Since $p * p \in Q$, we can cancel it from the left and obtain $r * s = s * s$. Since $s = 0 \in Q$ we can cancel it from the right and obtain $r = s$, a contradiction. This contradicts the assumption that $\lambda \not\leq \zeta$, so the proof is finished. □

THEOREM 2.12 *If* $\mathbf{S}$ *is a subdirectly irreducible mode of quasi-affine type, then the center* ζ *of* $\mathbf{S}$ *is nonzero, and all* ζ*-classes support quasi-affine subalgebras of* $\mathbf{S}$*.*

Proof: Lemma 2.10 shows that $(0:\mu) > 0$, and Lemma 2.11 proves that $\zeta = (0:\mu)$. Thus, the center is nonzero.

In the proof of Lemma 2.11 we saw that the ζ-class $Z = 0/\zeta$ is entirely contained in Q. This means that Z is closed under $*$ and that $x * y$ is a cancellative binary polynomial of the (abelian) subalgebra $\mathbf{Z}$ which commutes with itself and with all terms of $\mathbf{Z}$. The main result of [3] proves that $\mathbf{Z}$ is quasi-affine in these circumstances. Now suppose that Y is any other ζ-class. If $u \neq v \in Y$, then there is a polynomial $p_{uv} \in \mathrm{Pol}(\mathbf{S})$ such that $0 = p_{uv}(u) \neq p_{uv}(v)$ or the same with u and v switched. Since p_{uv} preserves congruences, and maps u or v to $0 \in Z$, it follows that $p_{uv}(Y) \subseteq Z$. Therefore $p_{uv} : \mathbf{Y} \to \mathbf{Z}$ is a homomorphism from $\mathbf{Y}$ to $\mathbf{Z}$ which separates u and v. The product of all such p_{uv}, $u \neq v \in Y$, is an embedding of $\mathbf{Y}$ into a quasi-affine algebra. □

We prove nothing more about the structure of subdirectly irreducible modes. However, in [6] more is shown for finite subdirectly irreducible modes, which suggests possible directions to explore. For example, it is shown that Q, as defined above, is a single ζ-class when $\mathbf{S}$ is finite. Moreover, in the finite case, if $\mathbf{S}$ is of quasi-affine type then $\mathbf{S}/\zeta$ has a semilattice term. We have no proof or counterexample for these statements in the infinite case.

3 Applications

The monolith of a subdirectly irreducible mode of semilattice type is nonabelian. The monolith of a subdirectly irreducible mode of quasi-affine type is abelian but not strongly abelian. We have proved nothing about subdirectly irreducible modes of set type, but in [6] it is shown that the monolith of a finite subdirectly irreducible mode of set type is strongly abelian. Thus, the classification of subdirectly irreducible modes into three types reflects commutator properties.

An algebra is said to be **(congruence) neutral** if it satisfies the commutator congruence equation $[\alpha, \beta] = \alpha \wedge \beta$. We will call an algebra $\mathbf{A}$ **hereditarily neutral** if every subalgebra of $\mathbf{A}$ is neutral. We will call $\mathbf{D}$ a **divisor** of $\mathbf{A}$ if $\mathbf{D}$ is a homomorphic image of a subalgebra of $\mathbf{A}$. It is easy to show that an idempotent algebra $\mathbf{A}$ is hereditarily neutral if and only if it has no nontrivial abelian divisor. From this, and our earlier results, it is easy to deduce that a subdirectly irreducible mode is hereditarily neutral if and only if it has a semilattice term. For if $\mathbf{S}$ is subdirectly irreducible mode of set type or quasi-affine type which has monolith μ, and M is a nontrivial μ-class, then the algebra $\mathbf{M}$ is a nontrivial abelian divisor of $\mathbf{S}$. Thus, if $\mathbf{S}$ is hereditarily neutral it has semilattice type, and therefore a semilattice term.

Homomorphic images of hereditarily neutral algebras are again hereditarily neutral, so if $\mathbf{A}$ is a hereditarily neutral mode then every subdirectly irreducible homomorphic image of $\mathbf{A}$ has a semilattice term. But this does not imply that $\mathbf{A}$ itself has a semilattice term!

Example 3.1 We are going to define a kind of mode which, within this example, we will call "chain modes".

Let C be an ordered chain. Let $\vee$ and $\wedge$ denote the join and meet operations of this chain. For each $c \in C$ define a binary operation on C by $\mathbf{c}(x, y) := x \wedge (y \vee c)$. It is readily checked that the algebra $\mathbf{C} = \langle C; \mathbf{c}(x, y), (c \in C)\rangle$ is a mode. We call any mode that is term equivalent to a mode constructed in this way a **chain mode**.

If $c_0 \in C$ is a minimal element, then $\mathbf{c}_0(x, y)$ is a semilattice term of $\mathbf{C}$. Conversely, if C has no minimal element and t is any term of $\mathbf{C}$, then t is composed from finitely many $\mathbf{c}_i(x, y)$, $c_i \in \{c_1, \ldots, c_n\}$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{C}$ with universe $\{b \in C \mid b \leq c_i \text{ for all } i\}$. Then $\mathbf{B}$ is nontrivial and $\mathbf{B} \models \mathbf{c}_i(x, y) = x$ for $i = 1, \ldots, n$. Hence t is a projection operation on $\mathbf{B}$. In particular, t cannot be a semilattice term of $\mathbf{C}$. Thus, $\mathbf{C}$ has a semilattice term if and only if C has a minimal element.

The observation of the previous paragraph shows that every finitely generated subalgebra of a chain mode has a semilattice term. This implies that if $\mathbf{C}$ is a chain mode, then every algebra in $\mathsf{HSP}_{fin}(\mathbf{C})$ is neutral. In particular, chain modes are hereditarily neutral. But if the underlying chain has no minimal element, then $\mathbf{C}$ does not have a semilattice term.

In contrast to this example, we will see that any hereditarily neutral mode *of finite signature* has a semilattice term.

LEMMA 3.2 *Let* $\mathbf{A}$ *be a mode of finite signature. Then* $\mathbf{A}$ *has a binary term* $x \wedge y$ *which interprets as a semilattice operation in each 2-element nonabelian divisor of* $\mathbf{A}$.

Proof: Since $\mathbf{A}$ has finite signature, there are only finitely many 2-element nonabelian divisors of $\mathbf{A}$ up to isomorphism: $\mathbf{D}_1, \ldots, \mathbf{D}_n$. Each such divisor is simple, hence subdirectly irreducible, and has the property that the monolith (which is the total congruence) is nonabelian. Therefore, it cannot be that any $\mathbf{D}_i$ is of quasi-affine type. If some $\mathbf{D}_i$ was of set type, then $\mathbf{D}_i$ would be term equivalent to a set. This cannot happen, because sets are abelian. Thus each $\mathbf{D}_i$ has semilattice type, and this implies that each $\mathbf{D}_i$ has a semilattice term.

Now suppose that $m_i(x, y)$ is a term in the language of $\mathbf{A}$ which interprets as a semilattice operation in $\mathbf{D}_i$. We describe a way to construct a single term $x \wedge y$ which interprets as a semilattice operation in every $\mathbf{D}_i$. First, given two binary terms $b(x, y)$ and $c(x, y)$, define

$$(b \diamond c)(x, y) := b(c(x, y), c(y, x)).$$

Notice that if $b(x, y)$ interprets as a semilattice operation on $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ and $c(x, y)$ interprets as a semilattice operation on $\mathbf{C} \in \mathcal{V}(\mathbf{A})$, then

$$\mathbf{C} \models (b \diamond c)(x, y) = b(c(x, y), c(y, x)) = b(c(x, y), c(x, y)) = c(x, y),$$

and

$$\mathbf{B} \models (b \diamond c)(x, y) = b(c(x, y), c(y, x)) = c(b(x, y), b(y, x)) = c(b(x, y), b(x, y)) = b(x, y).$$

Thus $(b \diamond c)(x, y)$ interprets as a semilattice operation on both $\mathbf{B}$ and $\mathbf{C}$. From this observation it is clear that

$$x \wedge y := (m_1 \diamond (m_2 \diamond \cdots (m_{n-1} \diamond m_n) \cdots))(x, y)$$

interprets as a semilattice operation on each $\mathbf{D}_i$. □

THEOREM 3.3 *If* $\mathbf{A}$ *is a hereditarily neutral mode of finite signature, then* $\mathbf{A}$ *has a semilattice term.*

Proof: Let $\{\mathbf{S}_i \mid i \in I\}$ be a representative set of subdirectly irreducible homomorphic images of $\mathbf{A}$. Each $\mathbf{S}_i$ is hereditarily neutral, and therefore is of semilattice type. For each i, let $\mathbf{M}_i$ be the subalgebra if $\mathbf{S}_i$ which is supported by the unique nontrivial congruence class of the monolith. Each $\mathbf{M}_i$ is a 2-element nonabelian divisor of $\mathbf{A}$, so Lemma 3.2 guarantees the existence of a term $x \wedge y$ which interprets as a semilattice operation in each $\mathbf{M}_i$. The proof of Theorem 2.3 shows that *any* essentially binary term which is noncancellative on $\mathbf{M}_i$ is a semilattice term of $\mathbf{S}_i$; therefore $x \wedge y$ is a semilattice term for each $\mathbf{S}_i$. Since $\mathbf{A}$ is a subdirect product of the $\mathbf{S}_i$, it follows that $x \wedge y$ is a semilattice term for $\mathbf{A}$. □

COROLLARY 3.4 *For a variety of modes,* $\mathcal{V}$*, the following conditions are equivalent.*

(1) $\mathcal{V}$ *has no abelian algebras.*

(2) *All algebras in* $\mathcal{V}$ *are neutral.*

(3) $\mathcal{V}$ *has a semilattice term.*

Proof: Since $\mathcal{V}$ is idempotent, $\mathcal{V}$ has no abelian algebras if and only if every member is hereditarily neutral, which holds if and only if every member is neutral since $\mathcal{V}$ is closed under the formation of subalgebras. Therefore (1) $\Longleftrightarrow$ (2) for any idempotent variety. The implication (3) $\Longrightarrow$ (2) follows from the fact that semilattices are neutral, and expansions of neutral algebras are neutral.

To finish, we assume (2) and derive (3). Assume that $\mathcal{V}$ is a neutral variety of modes. Neutrality is equivalent to a Mal'cev condition, as is proved in Corollary 4.7 of [7], so $\mathcal{V}$ has a finite sequence of terms $t_1, \ldots, t_k$ which witness that $\mathcal{V}$ is congruence neutral. Let $\mathcal{U}$ be the variety generated by the reducts of $\mathcal{V}$-algebras to the operations $t_1, \ldots, t_k$. Then $\mathcal{U}$ is a variety of modes of finite signature, which satisfies the Mal'cev condition for neutrality. It follows from Theorem 3.3 that each member of $\mathcal{U}$ has a semilattice term. In particular, the reduct of $\mathbf{F}_{\mathcal{V}}(3)$ to $t_1, \ldots, t_k$ has a semilattice term. This implies that $\mathbf{F}_{\mathcal{V}}(3)$ has a semilattice term, $x \wedge y$, constructible from $t_1, \ldots, t_k$. Since semilattices have an equational basis involving only three variables, and the term $x \wedge y$ will satisfy these laws throughout $\mathcal{V}$ if it is a semilattice term for $\mathbf{F}_{\mathcal{V}}(3)$, it follws that $x \wedge y$ is a semilattice term for $\mathcal{V}$. $\Box$

Next we turn our attention to the characterization of varieties of modes which have a Mal'cev term. We require the following preparatory lemma. If $\mathbf{A}$ is an algebra, $r = s$ is an equation in the language of $\mathbf{A}$ and $U \subseteq A$, then we say that $r = s$ **holds on** U provided that $r(x_1, \ldots, x_n) = s(x_1, \ldots, x_n)$ whenever all $x_i \in U$.

LEMMA 3.5 *Let $\mathbf{A}$ be a mode and $U \subseteq A$ be a subset. If $r = s$ holds on U, then $r = s$ holds on the subalgebra generated by U.*

Proof: Let $\mathbf{B}$ be the subalgebra generated by U. The assumption that $r = s$ holds on U means that the functions $r, s : U^n \to A$ agree. For $0 \leq i \leq n-1$, if

$$r, s : B^i \times U^{n-i} \to A$$

agree, then for $\mathbf{b} \in B^i$ and $\mathbf{u} \in U^{n-i-1}$ the equalizer of the endomorphisms

$$r(\mathbf{b}, x, \mathbf{u}), s(\mathbf{b}, x, \mathbf{u}) : \mathbf{A} \to \mathbf{A}$$

contains U, and therefore it contains $\mathbf{B}$. Hence if r and s agree on $B^i \times U^{n-i}$, then r and s agree on $B^{i+1} \times U^{n-i-1}$. By induction, $r = s$ holds on $\mathbf{B}$. $\Box$

THEOREM 3.6 *A variety of modes has a Mal'cev term if and only if it contains no algebra term equivalent to a 2-element set or a 2-element semilattice.*

Proof: A variety with a Mal'cev term cannot contain an algebra term equivalent to a 2-element set or 2-element semilattice, since these algebras do not have Mal'cev terms.

Assume that $\mathcal{V}$ is a variety of modes which contains no 2-element set or 2-element semilattice. This implies that there is no clone homomorphism from the clone of $\mathcal{V}$ onto the clone of the 2-element set or onto the clone of the 2-element semilattice. In particular, there is no clone homomorphism from the clone of $\mathcal{V}$ into the clone of the 2-element semilattice. Since the variety of semilattices is locally finite, a compactness argument proves that some finitely generated subclone of the clone of $\mathcal{V}$ has no homomorphism into the clone of the 2-element semilattice. This statement is equivalent to the statement that $\mathcal{V}$ satisfies an idempotent Mal'cev condition which fails in the variety of semilattices.

Now we are in a position to invoke Theorem 4.10 of [7], which states that in any variety satisfying an idempotent Mal'cev condition which fails in the variety of semilattices it is the case that abelian algebras are affine. Let $\mathcal{A}$ be the subvariety generated by the abelian algebras of $\mathcal{V}$. The free algebras in $\mathcal{A}$ are abelian, and so affine, and this implies that $\mathcal{A}$ is an affine variety. Let $m(x, y, z)$ be a Mal'cev term for $\mathcal{A}$. To finish the proof of this theorem we will prove that $m(x, y, z)$ is a Mal'cev term for $\mathcal{V}$, and that in fact $\mathcal{A} = \mathcal{V}$.

Choose any $\mathbf{S} \in \mathcal{V}$ which is subdirectly irreducible with monolith μ, and let M be a nontrivial μ-class. The subalgebra $\mathbf{M}$ supported by M cannot be term equivalent to a set or to a 2-element semilattice, because we have assumed that $\mathcal{V}$ contains no such algebras. It follows that every subdirectly irreducible $\mathbf{S} \in \mathcal{V}$ is of quasi-affine type. Thus each algebra $\mathbf{M}$ supported by a monolith class is abelian, hence belongs to $\mathcal{A}$, and this implies that $\mathbf{M} \models m(x, y, y) = x = m(y, y, x)$ for any such $\mathbf{M}$. We will use Lemma 3.5 to lift these equations from the monolith of a subdirectly irreducible algebra to the whole algebra.

If there is some algebra in $\mathcal{V}$ for which $m(x, y, z)$ is not a Mal'cev operation, then there is an algebra $\mathbf{A} \in \mathcal{V}$ generated by a 2-element set $\{a, b\}$ such that at least one of the following inequalities holds: $m(a, a, b) \neq b, m(b, b, a) \neq a, m(b, a, a) \neq b$, or $m(a, b, b) \neq a$. Factor $\mathbf{A}$ by a congruence θ which is maximal for the property that at least one of these four inequalities remains an inequality in the quotient $\mathbf{S} = \mathbf{A}/\theta$. Then $\mathbf{S}$ is subdirectly irreducible and generated by $\bar{a} = a/\theta$ and $\bar{b} = b/\theta$. The maximality of θ implies that all four inequalities become equalities in $\mathbf{S}/\mu$, where μ denotes the monolith $\mathbf{S}$. That is, Mal'cev's equations $m(x, y, y) = x = m(y, y, x)$ hold on $U = \{\bar{a}/\mu, \bar{b}/\mu\} \subseteq S/\mu$. But U generates $\mathbf{S}/\mu$. In this situation Lemma 3.5 proves that $\mathbf{S}/\mu \models m(x, y, y) = x = m(y, y, x)$, and so $\mathbf{S} \models m(x, y, y) \equiv_\mu x \equiv_\mu m(y, y, x)$. Therefore, $m(x, y, z)$ is a Mal'cev operation modulo μ, and also is a Mal'cev operation on μ-classes by the arguments in the fourth paragraph of this proof.

Let $m'(x, y, z) = m(x, m(x, y, y), m(x, y, z))$. Clearly, from this definition, the equation $m = m'$ holds in any algebra where $m(x, y, z)$ is a Mal'cev operation. However, in addition we have

$$m'(\bar{a}, \bar{b}, \bar{b}) = m(\bar{a}, m(\bar{a}, \bar{b}, \bar{b}), m(\bar{a}, \bar{b}, \bar{b})) = \bar{a}$$

since $\bar{a} \equiv_\mu m(\bar{a}, \bar{b}, \bar{b})$ and m is a Mal'cev operation on μ-classes. Similarly $m'(\bar{b}, \bar{a}, \bar{a}) = \bar{b}$. This shows that $m'(x, y, y) = x$ holds on $V = \{\bar{a}, \bar{b}\} \subseteq S$. Since V generates $\mathbf{S}$, this equation holds on $\mathbf{S}$.

Let $M(x, y, z) = m'(m'(x, y, z), m'(y, y, z), z)$. The same arguments as in the last paragraph show that $M = m'$ in any algebra where $m'(x, y, z)$ is a Mal'cev operation, and that $M(y, y, x) = x$ holds on V, and therefore on $\mathbf{S}$. But we also have

$$\mathbf{S} \models M(x, y, y) = m'(m'(x, y, y), m'(y, y, y), y) = m'(x, y, y) = x.$$

This proves that $M(x, y, z)$ is a Mal'cev operation on $\mathbf{S}$.

Since $\mathbf{S}$ is a mode, the Mal'cev term operation $M(x, y, z)$ commutes with all term operations of $\mathbf{S}$. This implies that $\mathbf{S}$ is affine, and so is a member of $\mathcal{A}$. But now we have a contradiction: we have shown that $\mathbf{S} \in \mathcal{A}$, that $m(x, y, z)$ is a Mal'cev term for all algebras in $\mathcal{A}$, but that $m(x, y, z)$ is not a Mal'cev term for $\mathbf{S}$. This contradiction arose from the assumption that $m(x, y, z)$ is not Mal'cev on some algebra in $\mathcal{V}$, so this assumption was false. $\square$

References

[1] D. Hobby and R. McKenzie, *The Structure of Finite Algebras*, Contemporary Mathematics v. 76, American Mathematical Society, 1988.

[2] K. A. Kearnes, *Idempotent simple algebras*, in "Logic and Algebra" (Proceedings of the Magari Memorial Conference, Siena), Marcel Dekker, New York, 1996.

[3] K. A. Kearnes, *A quasi-affine representation*, Internat. J. Algebra Comput. **5** (1995), 673–702.

[4] K. A. Kearnes, *Semilattice modes I: the associated semiring*, Algebra Universalis **34** (1995), 220–272.

[5] K. A. Kearnes, *Semilattice modes II: the amalgamation property*, Algebra Universalis **34** (1995), 273–303.

[6] K. A. Kearnes, *The structure of finite modes*, in preparation.

[7] K. A. Kearnes and Á. Szendrei, *The relationship between two commutators*, to appear in Internat. J. Algebra Comput.

[8] K. A. Kearnes and Á. Szendrei, *The classification of entropic minimal clones*, to appear in Discuss. Math. Algebra Stochastic Methods.

[9] A. Romanowska and B. Roszkowska, *Representations of n–cyclic groupoids*, Algebra Universalis **26** (1989), 7–15.

[10] Á. Szendrei, *Clones in Universal Algebra*, Séminaire de Mathématiques Supérieures **99**, Les Presses de L'Université de Montreal, 1986.

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