Jónsson posets and unary Jónsson algebras

KEITH A. KEARNES AND GREG OMAN

ABSTRACT. We show that if $\mathbf{P}$ is an infinite poset whose proper order ideals have cardinality strictly less than $|P|$, and κ is a cardinal number strictly less than $|P|$, then $\mathbf{P}$ has a principal order ideal of cardinality at least κ. We apply this result to characterize the possible sizes of unary Jónsson algebras.

1. Introduction

If $(P, \leq)$ is a finite poset, then the average size of a principal order ideal is the same as the average size of a principal order filter. Namely, it is the number of pairs in the relation $\leq$ divided by the number of elements in P. The situation is more asymmetric for infinite posets: If λ is an infinite cardinal and $[\lambda]^{<\omega}$ is the set of finite subsets of λ, then the principal order ideals of $([\lambda]^{<\omega}, \subseteq)$ are finite (so, they are very small) while the principal order filters each have size λ (which is as large as possible).

In this note, we present a nontrivial connection between the sizes of principal order ideals and principal order filters. We begin with the following definition.

Definition 1.1. Let $\kappa \leq \lambda$ be cardinals with λ infinite. A poset $\mathbf{P} = (P, \leq)$ is a (κ, λ)-*Jónsson poset* if and only if

(1) $|P| = \lambda$,
(2) any principal order ideal of $\mathbf{P}$ has size $< \kappa$, and
(3) the complement of any principal order filter of $\mathbf{P}$ has size $< \lambda$.

We shall prove that a (κ, λ)-Jónsson poset exists if and only if $\kappa = \lambda$. This means that, while it is easy to construct infinite posets whose principal order ideals are small and whose principal order filters are large in cardinality, if we insist that the principal order filters be large in the sense of having a small complement, then the least bounds on the sizes of the principal order ideals and order filters must be nearly the same. This result is easily seen to be equivalent to the statement that if $\mathbf{P}$ is an infinite poset whose proper order ideals have cardinality strictly less than $|P|$, and κ is a cardinal number strictly less than $|P|$, then $\mathbf{P}$ has a principal order ideal of size at least κ.

The terminology that we have selected for the posets we study is motivated by an application to the problem of determining the possible sizes of unary Jónsson algebras. We say that an infinite algebra $\mathbf{A}$ has *the Jónsson property* if $|B| < |A|$ whenever B is a proper subuniverse of $\mathbf{A}$. Recall that a *Jónsson*

2010 *Mathematics Subject Classification*: Primary: 06A06; Secondary: 08A60, 03E04.
Key words and phrases: Jónsson algebra, Jónsson poset, unary algebra.

algebra is an algebra with the Jónsson property, which is defined in a language with countably many function symbols. *Jónsson's Problem* is the problem of determining for which infinite cardinals λ there exists a Jónsson algebra of size λ. If there is no Jónsson algebra of size λ, then λ is called a *Jónsson cardinal*. The study of Jónsson's Problem is part of the investigation of large cardinals. It is known, for example, that every measurable cardinal is a Jónsson cardinal. On the other hand, ZFC + $(V = L)$ (the axiom of constructibility) implies that there are no Jónsson cardinals. We refer the reader to Jech [2] for results on Jónsson cardinals and to Coleman [1] for a survey of Jónsson algebras.

We employ our result on Jónsson posets to answer the following three questions:

(1) For which pairs of cardinals (μ, λ) is there an algebra of size λ with the Jónsson property, defined in a language with μ-many *unary* function symbols?
(2) For which pairs of cardinals (μ, λ) from the preceding question can the algebra be constructed so that all unary operations commute with one another?
(3) For which pairs of cardinals (μ, λ) from question (1) can the algebra be constructed so that the deletion of any single operation from the signature results in a loss of the Jónsson property?

2. Jónsson posets

Recall from the Introduction that, if $\kappa \leq \lambda$ are cardinals with λ infinite, then $\mathbf{P} = (P, \leq)$ is a (κ, λ)-*Jónsson poset* if

(1) $|P| = \lambda$,
(2) any principal order ideal of $\mathbf{P}$ has size $< \kappa$, and
(3) the complement of any principal order filter of $\mathbf{P}$ has size $< \lambda$.

We will soon show that a (κ, λ)-Jónsson poset exists exactly when $\kappa = \lambda$. First, we introduce some notation and terminology. If $(P, \leq)$ is a poset and $p \in P$, then we write $(p]$ for the principal order ideal generated by p (that is, $\{x \in P : x \leq p\}$) and $[p)$ for the principal order filter generated by p (that is, $\{x \in P : p \leq x\}$). If δ is a cardinal, we call a set "δ-small" if it has size $< \delta$. A subset of a set X will be called "δ-large" if its complement in X is δ-small.

Theorem 2.1. *There exists a* (κ, λ)*-Jónsson poset if and only if* $\kappa = \lambda$.

Proof. It is easy to see that there exist (λ, λ)-Jónsson posets for any infinite λ. Indeed, simply take $\mathbf{P} := (\lambda, \in)$, where $\in$ is membership.

To establish the other implication, assume by way of contradiction that $\mathbf{P} = (P, \leq)$ is a (κ, λ)-Jónsson poset and that $\kappa < \lambda$. Then the principal order ideals of $\mathbf{P}$ are κ-small and the principal order filters of $\mathbf{P}$ are λ-large.

Claim 2.2. $\mathrm{cf}(\lambda) \leq \kappa < \lambda$.

We begin by showing that $\mathbf{P}$ does not have maximal elements. If instead $p \in P$ is maximal, then the principal order filter $[p)$ is a singleton. But then the complement of $[p)$ has size λ, contradicting the assumption that $\mathbf{P}$ is a (κ, λ)-Jónsson poset.

We now define a well-ordered sequence $p_0 < p_1 < \cdots < p_\beta < \cdots$ in $\mathbf{P}$ to be *unbounded* if no element of $\mathbf{P}$ lies above all elements of the sequence, i.e., if the sequence is contained in no principal order ideal. Note that unbounded sequences exist and have order type equal to a infinite limit ordinal, since $\mathbf{P}$ has no maximal elements. Moreover, if $(p_\beta)_{\beta<\alpha}$ is an unbounded sequence in $\mathbf{P}$ and (α_i) is cofinal in α, then clearly (p_{α_i}) is also an unbounded sequence in $\mathbf{P}$. Hence, if $(p_\beta)_{\beta<\alpha}$ is an unbounded sequence in $\mathbf{P}$ having shortest length α, then α must be a cardinal and must be regular.

For each $p \in P$, call the complement of the principal order filter $[p)$ the *p-layer*. (More explicitly, the p-layer is $L(p) := \{x \in P : x \ngeq p\}$.) It follows easily by definition of $L(p)$ that

$$p_1 \leq p_2 \text{ implies } L(p_1) \subseteq L(p_2). \tag{2.1}$$

As $L(p)$ is the complement of a principal order filter, we deduce:

$$|L(p)| < \lambda \text{ for every } p \in P. \tag{2.2}$$

Note also that

$$\bigcup_{\beta<\alpha} L(p_\beta) = P, \tag{2.3}$$

since if $x \in P - \bigcup_{\beta<\alpha} L(p_\beta)$, then x would lie above all members of the unbounded sequence $(p_\beta)_{\beta<\alpha}$. Because the λ-element set P is the union of α-many λ-small sets, we conclude that

$$\mathrm{cf}(\lambda) \leq \alpha. \tag{2.4}$$

Since $\mathbf{P}$ is a (κ, λ)-Jónsson poset, each principal order ideal $(p_\beta]$ has size $< \kappa$. We claim that

$$\alpha \leq \kappa. \tag{2.5}$$

Suppose instead that $\kappa < \alpha$. Since $\kappa = |\{p_i : i < \kappa\}| \leq |(p_\kappa]|$ and the principal order ideals of $\mathbf{P}$ are κ-small, we get $\kappa \leq |(p_\kappa]| < \kappa$, a contradiction. Putting (2.4) and (2.5) together, we see that

$$\mathrm{cf}(\lambda) \leq \alpha \leq \kappa < \lambda, \tag{2.6}$$

and the proof of Claim 2.2 is complete.

Note from (2.6) above that λ is a singular cardinal. Since λ is infinite, we conclude that

$$\lambda \geq \aleph_\omega. \tag{2.7}$$

We now finish the proof of Theorem 2.1. Recalling from above that $(p_\beta)_{\beta<\alpha}$ is an unbounded sequence of least length α, we let λ_β denote the cardinality of the p_β-layer $L(p_\beta)$. Then (2.1) – (2.3) imply that $\lambda_\beta < \lambda$ for all β and the supremum of the λ_β's is λ. By thinning the sequence $(p_\beta)_{\beta<\alpha}$ if necessary, we may assume that $(\lambda_\beta)_{\beta<\alpha}$ is a strictly increasing sequence of infinite cardinals with supremum λ. (Here we are relying on (2.7) to ensure that the λ_β's can be chosen to be infinite.) After thinning, fix the unbounded sequence $(p_\beta)_{\beta<\alpha}$ until the end of the proof.

We now choose subsets X_β of P for $\beta < \alpha$ so that the following are true:

(i) $X_0 = \varnothing$.
(ii) If $\beta = \gamma + 1$, then $L(p_\gamma) \subseteq X_{\gamma+1} \subseteq L(p_{\gamma+1})$ and $|X_{\gamma+1}| = |L(p_\gamma)|^+$.
(iii) If β is limit, then $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$.

(The main points to remember are that the cardinals $\xi_\beta := |X_\beta|$ are strictly increasing with limit $\lambda = |P|$, and that ξ_β is regular when β is a successor ordinal. The construction of the X_β's is possible because the p_β-layers strictly increase in size.)

Let β^* be the least successor ordinal $\beta < \alpha$ such that $\xi_\beta = |X_\beta| > \kappa$ (which must exist if $\kappa < \lambda = \sup_{\beta<\alpha}(\xi_\beta)$). For each $x \in X_{\beta^*}$, the x-layer is λ-small, so $|L(x)| < |X_\beta|$ for large enough β. Let $\beta(x)$ be the least successor ordinal $\beta > \beta^*$ such that $|L(x)| < |X_\beta|$ holds. The map $X_{\beta^*} \to \alpha \colon x \mapsto \beta(x)$ is a function from a set X_{β^*} of regular cardinality ξ_{β^*} to a set α of smaller cardinality. (Recall that $\xi_{\beta^*} > \kappa \geq \alpha$.) Necessarily there is a subset $Y \subseteq X_{\beta^*}$ of size ξ_{β^*} such that $x \mapsto \beta(x)$ is constant on Y. Let $\gamma < \alpha$ be that constant value. For each $y \in Y$, we have that

$$|L(y)| < |X_{\beta(y)}| = |X_\gamma|. \tag{2.8}$$

The last equality follows since $\beta(y) = \gamma$ by definition of γ. The set $X_\gamma - \bigcup_{y\in Y} L(y)$ cannot be empty since $\xi_\gamma = |X_\gamma|$ is regular (since $\gamma = \beta(x)$ for some x and $\beta(x)$ is by definition a successor ordinal), $|L(y)|$ is ξ_γ-small (by (2.8) above), and $|Y| = \xi_{\beta^*} < \xi_\gamma$ (since $\gamma = \beta(y)$ for some y, $\beta^* < \beta(y)$, and the cardinals ξ_i are strictly increasing). But if $z \in X_\gamma - \bigcup_{y\in Y} L(y)$, then z lies above every element of Y. Hence the principal order ideal $(z]$ has size at least $|Y| = \xi_{\beta^*} > \kappa$. This contradiction concludes the proof. □

Remark 2.3. It is now possible to eliminate κ from the definition of a (κ, λ)-Jónsson poset: Call an infinite poset $\mathbf{P}$ a *Jónsson poset* (without modifying parameters) if the complement of any principal order filter has size $< |P|$, equivalently, if any proper order ideal of $\mathbf{P}$ is $|P|$-small. Theorem 2.1 then becomes a statement about the size of principal order ideals in such posets, namely that if $\mathbf{P}$ is a Jónsson poset and $\kappa < |P|$, then $\mathbf{P}$ has a principal order ideal of size at least κ.

Example 2.4. (Some Jónsson posets.)

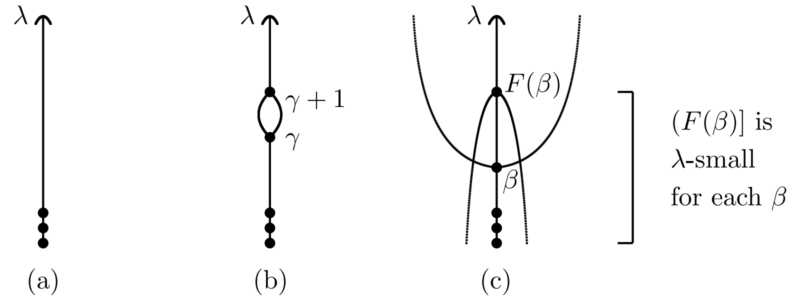

Figure 1. Some Jónsson posets of type (λ, F)

Let λ be a cardinal and let $F\colon \lambda \to \lambda$ be an order-preserving function (i.e., $\alpha \leq \beta$ implies $F(\alpha) \leq F(\beta)$). Extend $(\lambda, \in)$ to a new poset $\mathbf{P} = (P, \leq)$ by adding new elements and comparabilities in an arbitrary way as long as the poset remains "of type (λ, F)", which we define to mean:

(i) the elements of λ form a cofinal sequence in $\mathbf{P}$,
(ii) for each $\beta < \lambda$, $P = [\beta) \cup (F(\beta)]$, and
(iii) the principal ideal $(F(\beta)]$ is λ-small.

It is not hard to see that, for any cardinal λ and order-preserving function F, a poset of type (λ, F) is Jónsson.

The smallest possibility for F is

$$F(\beta) = \begin{cases} \gamma & \text{if } \beta = \gamma + 1; \\ \beta & \text{if } \beta \text{ is either 0 or a limit.} \end{cases}$$

The only poset of type (λ, F) for this F is $(\lambda, \in)$. This poset is indicated in Figure 1 (a).

If we alter this smallest F to one which, for a particular successor ordinal $\gamma + 1$, satisfies $F(\gamma + 1) = \gamma + 1$ instead of $F(\gamma + 1) = \gamma$, then we obtain a poset like the one in Figure 1 (b). Here we have fattened $(\lambda, \in)$ by adding a λ-small set of arbitrarily-ordered elements between γ and $\gamma + 1$.

The general poset of type (λ, F) is suggested by Figure 1 (c). The restrictions depicted there must hold for each β.

If $\mathbf{P}$ is any Jónsson poset of regular cardinality λ, then an unbounded sequence in $\mathbf{P}$ must have length at least λ (by (2.6)). The first λ-many elements of such a sequence will generate an ideal of size at least λ, which cannot be proper, hence the sequence must be cofinal. Let $(p_\beta)_{\beta<\lambda}$ be a cofinal sequence in $\mathbf{P}$ of length λ. For any $\beta < \lambda$, the p_β-layer has size $< \lambda$. Since λ is regular and $(p_\beta)_{\beta<\lambda}$ is cofinal in $\mathbf{P}$, there is a least $F(\beta)$ such that the element $p_{F(\beta)}$ majorizes the p_β-layer. Taking the sequence $(p_\beta)_{\beta<\lambda}$ to be a copy of the poset $(\lambda, \in)$ and endowing it with the function F just described, we see that any Jónsson poset of regular cardinality λ is of type (λ, F) for some order-preserving F. Moreover, items (i), (ii), and (iii) from above hold for F.

Remark 2.5. The statements made in the previous paragraph need not be true if λ is singular. We claim that if λ is singular, then there exists a Jónsson poset $\mathbf{P}$ of cardinality λ but with no cofinal sequence of length λ. To see this, let λ be an infinite cardinal and S be a set of ordinals cofinal in λ. For $i \in \lambda$, let $\bar{i}$ be the least element s of S (with respect to the usual $\in$-order on the ordinals) such that $i \in s$. Now define an order $<$ on λ by $i < j$ if and only if $\bar{i} \in \bar{j}$. One checks easily that $\mathbf{P} := (\lambda, \leq)$ is a Jónsson poset with the property that any strictly increasing sequence has size at most $|S|$. Our claim follows.

3. Unary Algebras with the Jónsson Property

Say that $\mathbf{A}$ is a *μ-unary algebra* if $\mathbf{A}$ is an algebra defined in a language with μ-many function symbols, all unary. If λ is infinite and $\mu \geq \lambda$, then it is easy to see that there is a μ-unary algebra of size λ with the Jónsson property. Indeed, let X be a set of size λ. For $c \in X$, let $f_c\colon X \to X$ be defined by $f_c(x) := c$ for all $x \in C$. Now interpret λ-many of the function symbols as the constant functions f_c, where c ranges over X, and the remaining function symbols (if any) arbitrarily. The resulting algebra has no proper subuniverses and so trivially has the Jónsson property. The next result concerns the existence of μ-unary algebras of size λ with the Jónsson property when $\mu < \lambda$.

Theorem 3.1. *If $\mu < \lambda$ are infinite cardinals, then the following are equivalent.*

(i) *There is a μ-unary algebra of size λ with the Jónsson property.*
(ii) *There is a (μ^+, λ)-Jónsson poset.*

Moreover, if (i) or (ii) holds, then $\lambda = \mu^+$.

Proof. [(i)⇒(ii)]
Assume that $\mathbf{A}$ is a μ-unary algebra of size λ with the Jónsson property. Let $\mathbf{P}$ be the poset of cyclic (i.e. 1-generated) subuniverses of $\mathbf{A}$ ordered by inclusion. Since every cyclic subuniverse has size at most μ and A is the union of all cyclic subuniverses, it is easy to see that $|P| = \lambda$. We now show that every principal order ideal of $\mathbf{P}$ has size at most μ. Let $\varphi\colon A \to P$ be the surjective function $a \mapsto \langle a \rangle$. Consider an arbitrary principal order ideal $(\langle a \rangle]$. Note that $\langle x \rangle \in (\langle a \rangle]$ iff $\langle x \rangle \subseteq \langle a \rangle$ iff $x \in \langle a \rangle$. Thus $\varphi\big|_{\langle a \rangle}\colon \langle a \rangle \to (\langle a \rangle]$ is a surjection. We conclude that $|(\langle a \rangle]| \leq |\langle a \rangle| \leq \mu$.

Now let $p \in P$ be arbitrary. We must show that the complement of $[p)$ has size less than λ. Note that the complement of $[p)$ is just the p-layer $L(p)$ of $\mathbf{P}$. Furthermore, as $\mathbf{A}$ is a unary algebra, $\varphi^{-1}(L(p))$ is a proper subuniverse of $\mathbf{A}$. Thus by the Jónsson property it has cardinality $< \lambda$. Hence the p-layer $L(p) = \varphi(\varphi^{-1}(L(p)))$ has cardinality $< \lambda$, and we have established the implication.

[(ii)⇒(i)]

Let $(P, \leq)$ be a (μ^+, λ)-Jónsson poset. Every principal order ideal has size $< \mu^+$, hence has size $\leq \mu$. Therefore, for each $p \in P$ there is a surjective function $f_p \colon \mu \to (p]$. Define a sequence of unary functions $(F_\nu)_{\nu<\mu}$ by $F_\nu(p) = f_p(\nu)$. This produces a μ-unary algebra $\mathbf{A} = (P, \{F_\nu : \nu < \mu\})$ whose subuniverses are exactly the order ideals of $(P, \leq)$. If $B \subseteq P$ is a proper subuniverse of $\mathbf{A}$, then it is a proper order ideal of $(P, \leq)$. But then B is disjoint from some principal order filter of $(P, \leq)$, whence B is λ-small. This establishes the Jónsson property for $\mathbf{A}$.

If (ii) holds, then Theorem 2.1 implies that $\lambda = \mu^+$. This completes the proof. □

Corollary 3.2. *If μ is infinite, then any μ-unary algebra with the Jónsson property has size at most μ^+. Moreover, there exist μ-unary algebras with the Jónsson property of size μ^+.*

Proof. The first assertion follows immediately from Theorem 3.1. As for examples, recall from the remarks at the beginning of Section 2 that $(\mu^+, \in)$ is a (μ^+, μ^+)-Jónsson poset. Theorem 3.1 implies that there exists a μ-unary algebra of size μ^+ with the Jónsson property. □

We now turn our attention to determining the possible sizes of commutative unary algebras with the Jónsson property (we say that a unary algebra $\mathbf{A}$ is *commutative* if its operations pairwise commute).

Proposition 3.3. *If $\mathbf{A}$ is a commutative μ-unary algebra of size λ with the Jónsson property, then $\aleph_0 \leq \lambda \leq \mu + \aleph_0$ and $\mu \neq 0$. Moreover, examples exist for any choice of cardinals satisfying $\aleph_0 \leq \lambda \leq \mu + \aleph_0$ and $\mu \neq 0$.*

Proof. We apply the phrase "Jónsson property" to infinite algebras only, so $\aleph_0 \leq \lambda$ holds. Any infinite 0-unary algebra is a structureless set, hence has proper subuniverses of the same cardinality. Thus we must have $\mu \neq 0$.

To prove the rest of the first statement, suppose by way of contradiction that there exists a commutative μ-unary algebra $\mathbf{A}$ of size λ with the Jónsson property and that $\mu + \aleph_0 < \lambda$. If μ is finite, expand the language so that it is countably infinite, either by introducing new functions symbols for each of the unary terms or by introducing $\aleph_0$-many new function symbols that interpret as the identity function. Now we are in a setting where we may apply Corollary 3.2, and it tells us that $\lambda = \mu^+$.

Partition the basic operations as $F := S \cup N$, where S is the set of surjective basic operations and N is the set of nonsurjective basic operations. Let $Y := \bigcup_{f \in N} f[A]$. Since the operations in F commute, $f[A]$ is a subuniverse of $\mathbf{A}$ for every $f \in F$. It follows that Y is the union of fewer than μ^+ subuniverses each of size less than μ^+. Since μ^+ is regular, we see that $|Y| < \mu^+$. Let $a \in A - Y$ be arbitrary, and let B be a subuniverse of $\mathbf{A}$ maximal with respect to containing Y and avoiding a. As $\mathbf{A}$ has the Jónsson property, it follows that

$$|B| < \mu^+. \tag{3.1}$$

Let $x \in A - B$ be arbitrary. The maximality of B implies that $a \in B \cup \langle x \rangle$. Since $a \notin B$, we deduce that $a \in \langle x \rangle$. Thus $a = t(x)$ for some term operation t. If $t = f_{i_1} \circ \cdots \circ f_{i_k}$ for basic operations $f_{i_j} \in F$, then each f_{i_j} must be from S, since all operations from N have range in the subuniverse Y and $t(x) = a \notin Y$. Thus, if T is the set of term operations composed of surjective basic operations, then we have shown that $A - B \subseteq \bigcup_{t \in T} t^{-1}(\langle a \rangle)$, equivalently

$$A = B \cup \bigcup_{t \in T} t^{-1}(\langle a \rangle). \tag{3.2}$$

Each term operation t is an endomorphism, so each set $t^{-1}(\langle a \rangle)$ is a subuniverse of $\mathbf{A}$, and there are at most $|T| \leq \mu$ of them. Hence (3.2) expresses A as a union of at most μ-many subuniverses. Since $|A| = \mu^+$ is regular, $|B| < \mu^+$ and $\mathbf{A}$ is Jónsson, we conclude that $A = t^{-1}(\langle a \rangle)$ for some $t \in T$. This yields $t[A] \subseteq \langle a \rangle$. However, since t is a composition of surjective functions, $A = t[A]$ $(\subseteq \langle a \rangle)$. We have arrived at the contradiction $\mu^+ = |A| \leq |\langle a \rangle| \leq \mu$, completing the proof of the first assertion of the proposition.

We give examples with $\aleph_0 \leq \lambda \leq \mu + \aleph_0$ and $\mu \notin \{0, 1\}$. (The case $\mu = 1$ and $\lambda = \aleph_0$ is described in Corollary 3.4.) Let G be an abelian group of cardinality λ that is μ-generated as a monoid. Since we do not assume that generators are distinct, such groups G exist for any $\aleph_0 \leq \lambda \leq \mu + \aleph_0$ and any $\mu \notin \{0, 1\}$. (See the proof of Lemma 4.2 for more detail.) For each generator g, let $f_g \colon G \to G$ be defined by $f_g(x) := gx$. The algebra $\mathbf{G} := (G, \{f_g\})$ has size λ, is μ-unary, and is commutative. Since it has no proper nontrivial subuniverses, it has the Jónsson property. □

We now recall from the Introduction that an infinite algebra is a Jónsson algebra provided it has the Jónsson property and is defined in a language with countably many function symbols.

Corollary 3.4. *Suppose that* $\mathbf{A}$ *is a unary Jónsson algebra. Then either* $|A| = \aleph_0$ *or* $|A| = \aleph_1$*. If* $\mathbf{A}$ *is commutative, then* $|A| = \aleph_0$*. Moreover, there exists a commutative* 1*-unary Jónsson algebra of size* $\aleph_0$ *and a* 2*-unary Jónsson algebra of size* $\aleph_1$ *(but no* 1*-unary Jónsson algebra of size* $\aleph_1$*).*

Proof. Let $\mathbf{A}$ be a μ-unary Jónsson algebra. If μ is finite, expand the language by adding $\aleph_0$-many unary function symbols which interpret as the identity on A. The resulting algebra is still Jónsson. Thus we may assume without loss of generality that $\mathbf{A}$ is $\aleph_0$-unary. Corollary 3.2 implies that $|A| \leq \aleph_1$. If $\mathbf{A}$ is commutative, then $|A| = \aleph_0$ by the previous proposition.

As for the examples, consider $\mathbf{A} := (\mathbb{N}, \{P\})$, where $P(n) := n - 1$ for $n > 0$ and $P(0)$ is defined arbitrarily. It is readily checked that $\mathbf{A}$ is Jónsson. For a 2-unary example of size $\aleph_1$, we refer the reader to Jónsson [3], pp. 128-129 (this example is attributed to Galvin). Note that there is no 1-unary Jónsson algebra of size $\aleph_1$, since any such algebra would be commutative. □

Remark 3.5. It is known that the algebras $\mathbf{A} := (\mathbb{N}, \{P\})$ (as defined in the proof of the previous corollary) are the *unique* 1-unary Jónsson algebras up to isomorphism (Corollary 4.2 of Oman [4]).

We conclude this section by giving a Jónsson-themed characterization of the structure $(\mathbb{N}, P, \leq)$, where $P : \mathbb{N} \to \mathbb{N}$ is the predecessor function on $\mathbb{N}$ (as defined in the proof of the previous corollary) and $\leq$ is the usual order relation.

Proposition 3.6. *Let $(X, \leq)$ be an infinite poset and let $f\colon X \to X$ be a function. Then $(X, f, \leq) \cong (\mathbb{N}, P, \leq)$ if and only if the following hold:*

(1) *$(X, \{f\})$ is a Jónsson algebra.*
(2) *$(X, \leq)$ is a Jónsson poset.*
(3) *For all $x, y \in X$: If $x \leq y$, then $f(x) \leq f(y)$.*
(4) *For all $x, y \in X$: If $f(x) < f(y)$, then $x < y$.*

Proof. We assume that $(X, f, \leq)$ satisfies (1)–(4). Remark 3.5 implies that there exists a pairwise-distinct enumeration of X, say $x_0, x_1, x_2, \ldots$ such that

$$f(x_n) = x_{n-1} \text{ for every } n > 0, \tag{3.3}$$

while $f(x_0) = x_k$ for some $k \in \mathbb{N}$. It will be demonstrated that $k = 0$. First, we claim

$$x_0 \leq x_1. \tag{3.4}$$

Since the complement of the principal order filter $[x_0)$ is finite, there exists $0 < m \in \mathbb{N}$ such that $x_0 \leq x_{1+m(k+1)}$. Condition (3) yields the inequality

$$f^{m(k+1)}(x_0) \leq f^{m(k+1)}(x_{1+m(k+1)}). \tag{3.5}$$

We conclude from (3.3) and the fact that $f(x_0) = x_k$ that $f^{m(k+1)}(x_0) = x_0$ and $f^{m(k+1)}(x_{1+m(k+1)}) = x_1$. Thus (3.5) is just $x_0 \leq x_1$, which was to be shown. We now establish that

$$x_n \leq x_{n+1} \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

We just proved that $x_0 \leq x_1$. Claim (3.6) follows by induction and Condition (4). Since $\leq$ is a partial order, we deduce that for all $m, n \in \mathbb{N}$: $x_m \leq x_n$ if and only if $m \leq n$. Recall above that $f(x_0) = x_k$. It remains to show that $k = 0$. Suppose by way of contradiction that $0 < k$. Then $x_0 \leq x_k$, and applying Condition (3), $f^k(x_0) \leq f^k(x_k)$. But this reduces to $x_1 \leq x_0$, and we have a contradiction to (3.4). □

4. Minimal unary algebras with the Jónsson property

If (A, F) has the Jónsson property and G is a collection of operations on A containing F, then (A, G) also has the Jónsson property. Call (A, F) *minimal* if (A, F) has the Jónsson property but (A, F') does not have the Jónsson property whenever $F' \subsetneq F$. In this section we consider the following question: For which cardinal pairs (μ, λ) (with λ infinite, as usual) does there exist a minimal μ-unary algebra $\mathbf{A}$ of size λ with the Jónsson property? We answer this question through a sequence of lemmas. Throughout the section, if $\mathbf{A}$ is an algebra and f is a basic operation, the algebra $\mathbf{A}_f^-$ denotes the algebra obtained from $\mathbf{A}$ by deleting f from the signature.

Lemma 4.1. *If $\mathbf{A} = (A, F)$ is an infinite (not necessarily unary) algebra with $|A| = \lambda$ and $|F| = \mu$ and $\lambda < \mu$, then for some $f \in F$ the algebra $\mathbf{A}_f^-$ has the same subuniverses as $\mathbf{A}$.*[1]

Hence, if $\mathbf{A}$ is a minimal μ-unary algebra of size λ which has the Jónsson property, then $\lambda \geq \mu$.

Proof. The second assertion of the lemma follows from the first, so we prove the first only.

Case 1. There exists some n-ary $f \in F$ such that for any $\mathbf{a} \in A^n$ there is some n-ary term $t_{\mathbf{a}}$, not involving f, for which $f(\mathbf{a}) = t_{\mathbf{a}}(\mathbf{a})$.

If $B \subseteq A$ is a subset closed under $F - \{f\}$, then it is easily seen that B is closed under f, too. For, if $\mathbf{a} \in B^n$ is arbitrary, then for the appropriate f-free term $t_{\mathbf{a}}$ we have $f(\mathbf{a}) = t_{\mathbf{a}}(\mathbf{a}) \in B$. This shows that subuniverses of $\mathbf{A}_f^-$ are subuniverses of $\mathbf{A}$ in Case 1.

Case 2. For every n-ary $f \in F$ there exists $\mathbf{a}_f \in A^n$ such that $f(\mathbf{a}_f) \neq t(\mathbf{a}_f)$ for any n-ary f-free term t.

Define $\varphi\colon F \to \bigcup_{n\in\omega}(A^n \times A)$ by $\varphi(f) := (\mathbf{a}_f, f(\mathbf{a}_f))$. If $f \neq g$ and

$$(\mathbf{a}_f, f(\mathbf{a}_f)) = \varphi(f) = \varphi(g) = (\mathbf{a}_g, g(\mathbf{a}_g)),$$

then $\mathbf{a}_f = \mathbf{a}_g$ and $f(\mathbf{a}_f) = g(\mathbf{a}_f)$, contrary to the fact that $f(\mathbf{a}_f) \neq t(\mathbf{a}_f)$ for any f-free term t. Hence φ is one-to-one, which shows that

$$\mu = |F| \leq |\bigcup_{n\in\omega}(A^n \times A)| = \lambda.$$

Hence Case 2 cannot arise when $\mu > \lambda$. This completes the proof. □

Lemma 4.2. *For any cardinal number $\mu \neq 0$, there exists a minimal commutative μ-unary algebra of size $\mu + \aleph_0$ which has the Jónsson property.*

[1] In fact, something stronger than this is true: If (A, F) is an infinite algebra with $|A| = \lambda$, then there is a subset $F_0 \subseteq F$ with $|F_0| \leq \lambda$ such that the powers $(A, F)^k$ and $(A, F_0)^k$ have the same subuniverses for every finite k.

Proof. We consider three cases.

Case 1. $\mu = 1$.
In this case, the algebra $(\mathbb{N}, \{P\})$ defined in Corollary 3.4 is such an example.

Case 2. $1 < \mu < \aleph_0$.
Let $n := \mu - 1$, and let G be the free abelian group of rank n (written multiplicatively). Fix a basis $\{b_1, b_2, \ldots, b_n\}$ for G, and let $b_{n+1} := (b_1 b_2 \cdots b_n)^{-1}$. Then $\{b_1, b_2, \ldots, b_n, b_{n+1}\}$ generates G as a monoid. For each i, $1 \leq i \leq n + 1 = \mu$, let $f_i : G \to G$ be defined by $f_i(g) := b_i g$ and consider the algebra $\mathbf{A} := (G, \{f_i : 1 \leq i \leq \mu\})$. Since the b_i generate G as a monoid, it is clear that $\mathbf{A}$ has no proper nontrivial subuniverses, whence is trivially Jónsson. If one deletes any f_i from the signature, then the cyclic subuniverse $\langle 1 \rangle$ is a proper, infinite subuniverse of $\mathbf{A}^-_{f_i}$. We conclude that $\mathbf{A}$ is minimal.

Case 3. $\aleph_0 \leq \mu$.
Let G be the free abelian group of rank μ, and let $\{b_i : i < \mu\}$ be a basis. Then the set $S := \{b_i : i < \mu\} \cup \{b_i^{-1} : i < \mu\}$ generates G as a monoid. As above, for $s \in S$, we define $f_s : G \to G$ by $f_s(g) := sg$. Also as above, one checks easily that the algebra $\mathbf{A} := (G, \{f_s : s \in S\})$ has no proper nontrivial subuniverses, hence has the Jónsson property. If one removes any f_s from the signature, then the cyclic subuniverse $\langle 1 \rangle$ is a proper, infinite subuniverse of $\mathbf{A}^-_{f_s}$ of size μ, and the proof is complete. □

Lemma 4.3. *If μ is infinite, then there exists a minimal μ-unary algebra of size μ^+ which has the Jónsson property.*

Proof. Assume μ is infinite, and consider the (μ^+, μ^+)-Jónsson poset $\mathbf{P} := (P, \leq) := (\mu^+, \in)$. For each $p \in P$, we let $f_p : \mu \to (p]$ be a surjection, and for $\nu < \mu$, we define $F_\nu : P \to P$ by $F_\nu(p) := f_p(\nu)$. Recall from Theorem 3.1 that $\mathbf{A} := (P, \{F_\nu : \nu < \mu\})$ is a μ-unary algebra of size μ^+ with the Jónsson property. Now let $T := \{t_i : i < \mu\}$ be a set indexed by μ which is disjoint from P. Let $B := T \cup P$, and for each $i < \mu$, define G_i on B as follows:

(1) $G_i(x) := t_i$ for all $x \in T \cup \mu$, and
(2) $G_i(y) := F_i(y)$ if $\mu \leq y < \mu^+$.

We claim that the algebra $\mathbf{B} := (B, \{G_i : i < \mu\})$ has the Jónsson property. Suppose B' is a subuniverse of $\mathbf{B}$ of size μ^+. Since $|T| = \mu$, it follows that $B' \cap P = B' \cap \mu^+$ has size μ^+, hence is cofinal in $P = (\mu^+, \in)$. Choose any $z \in B' \cap P$ such that $\mu \leq z$, and then choose any $\beta < z$. Recall from above that $f_z : \mu \to (z]$ is a surjection. Since $\beta < z$, it follows that there exists some $\gamma < \mu$ such that $f_z(\gamma) = \beta$. But by definition of the F_i and G_i above we get $\beta = f_z(\gamma) = F_\gamma(z) = G_\gamma(z)$. Since B' is closed under G_γ, and $z \in B'$, we conclude that $\beta \in B'$. This argument holds for any β majorized by some z in the cofinal subset $B' \cap P$ of $(\mu^+, \in)$, implying that $P = \mu^+ \subseteq B'$. It now follows from (1) above that $T \subseteq B'$ also, so $B' = B$. This proves that $\mathbf{B}$ is Jónsson.

Finally, for $i < \mu$, the set $B - \{t_i\}$ is a proper subuniverse of $\mathbf{B}^-_{G_i}$ having size μ^+, proving that $\mathbf{B}$ is minimal. □

Lemma 4.4. *If $1 < \mu \leq \aleph_0$, then there exists a minimal μ-unary algebra of size $\aleph_1$ which has the Jónsson property.*

Proof. Let β be the cardinal number satisfying $2 + \beta = \mu$, and let $\mathbf{A} = (A, \{f, g\})$ be a 2-unary Jónsson algebra (such as Galvin's example) of size $\aleph_1$. As above, we let T be a set of size β which is disjoint from A, and we let $B := T \cup A$. For $t \in T$ define $h_t \colon T \to T$ by $h_t(x) := t$ for all $x \in T$. Since, by Corollary 3.4, $(A, \{f\})$ is not Jónsson, there exists a proper f-subuniverse C of $\mathbf{A}$ of size $\aleph_1$. Let $c \in C$ be arbitrary. Now extend f to T by defining $f(t) := c$ for all $t \in T$. Analogously, since $(A, \{g\})$ is not Jónsson, there exists a proper g-subuniverse D of $\mathbf{A}$ of size $\aleph_1$. Let $d \in D$ be arbitrary. Extend g to T by defining $g(t) := d$ for all $t \in T$. Lastly, extend each h_t to B by defining $h_t(\alpha) := t$ for all $\alpha \in A$. We now show that the resulting algebra $\mathbf{B}$ is Jónsson. Indeed, we suppose by way of contradiction that E is a proper subuniverse of $\mathbf{B}$ of size $\aleph_1$. Then $E \cap A$ is a subuniverse of $\mathbf{A}$ of size $\aleph_1$. Since $\mathbf{A}$ is Jónsson, we deduce that $E \cap A = A$, that is, $A \subseteq E$. But since E is closed under each h_t, it follows that $T \subseteq E$ as well. Thus $E = B$, and we have reached a contradiction. We have shown that $\mathbf{B}$ is Jónsson. We now show that $\mathbf{B}^-_f$ is not Jónsson. Toward this end, consider $D \cup T$. It follows from our above definitions that $D \cup T$ is a proper subuniverse of $\mathbf{B}^-_f$ of size $\aleph_1$. We now claim that $\mathbf{B}^-_g$ is not Jónsson. This follows analogously by considering $C \cup T$. Finally, let $t \in T$. We must show that $\mathbf{B}^-_{h_t}$ is not Jónsson. This follows by considering $B - \{t\}$. Hence $\mathbf{B}$ is minimal, and the proof is complete. □

Putting these results together, we obtain the following theorem:

Theorem 4.5. *Let μ and λ be cardinals with μ nonzero. There exists a minimal μ-unary algebra of size λ with the Jónsson property if and only if*

$$\mu + \aleph_0 \leq \lambda \leq \mu^+ + \aleph_1.$$

Acknowledgment. The authors thank F. Wehrung for suggesting Claim 2.2 and contributing its proof.

References

[1] Coleman, E.: *Jónsson groups, rings, and algebras*, Irish Math. Soc. Bull. **36** (1996), 34–45.

[2] Jech, T.: *Set Theory (third millennium edition)*, Springer Monographs in Mathematics, New York, 2002.

[3] Jónsson, B.: *Topics in Universal Algebra*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[4] Oman, G.: *On elementarily κ-homogeneous unary structures*, Forum Math. **23** (2011), 791–802.

Keith A. Kearnes
Department of Mathematics, University of Colorado, Boulder CO 80309-0395, USA
e-mail: kearnes@euclid.colorado.edu
URL: http://spot.colorado.edu/~kearnes/

Greg Oman
Department of Mathematics, University of Colorado, Colorado Springs CO 80918, USA
e-mail: goman@uccs.edu
URL: http://www.uccs.edu/~faculty/goman/