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Congruence Permutable and Congruence 3-Permutable Locally Finite Varieties

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We give a tame congruence-theoretic characterization of congruence permutable and congruence 3-permutable locally finite varieties. © 1993 Academic Press, Inc.

1. INTRODUCTION

In 1954, A. I. Mal'cev published [5], which contains the result that all members of a variety of algebras have permuting congruences if and only if the variety satisfies the equations

$$p(x, x, y) \approx y \qquad \text{and} \qquad p(x, y, y) \approx x$$

with respect to some ternary term $p(x, y, z)$. This paper initiated the study of so-called *Mal'cev conditions* which postulate the existence of terms satisfying certain equations. Since Mal'cev's paper, many algebraic properties of varieties have been shown to be equivalent to Mal'cev conditions; e.g., congruence modularity, congruence distributivity, congruence n-permutability. One might say, with little exaggeration, that the general problem of classifying varieities has become synonymous with the study of Mal'cev conditions.

In the early 1980's, Ralph McKenzie and his student David Hobby developed a structure theory for finite algebras and locally finite varieties which they call *tame congruence theory*. In their book [4], they show that for locally finite varieties many of the properties known to be definable by Mal'cev conditions were equivalent to local structural properties holding for each member of the variety. Chapter 9 of [4] includes tame congruence-theoretic characterizations of congruence modularity, and congruence distributivity, as well as other well-known congruence conditions. What is important for us in this paper is that this chapter

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includes a tame congruence-theoretic characterization of the property that a locally finite variety is congruence n-permutable for some n. This chapter does not include a characterization of the locally finite varieties which are congruence n-permutable for any specific value of n, but a plausible characterization of the congruence permutable (= congruence 2-permutable) locally finite varieties is suggested in Exercise 8.8 (1).

The question of whether the condition suggested by Hobby and McKenzie characterizes congruence permutable locally finite varieties remained open until 1989. While studying a completely different question, P. Idziak proved that a finite algebra which fails to have permuting congruences, but generates a congruence modular variety, has a pair of non-permuting congruences α, β which cover their meet: $\alpha \wedge \beta \prec \alpha, \beta$. (McKenzie later removed the congruence modularity hypothesis.) M. Valeriote and R. Willard saw a connection between this result and the Hobby–McKenzie problem and soon produced a proof that the condition suggested in 8.8 (1) of [4] characterizes congruence permutable varieties. Their proof appears in [8].

The Idziak–McKenzie result, on which the arguments of Valeriote and Willard hinge, is false for n-permuting congruences for any value of $n > 2$. For example, the semilattice $\langle \mathscr{P}(\{x, y\}); \cap \rangle$ has congruences α, β which fail to 3-permute, but has no such α and β satisfying $\alpha \wedge \beta \prec \alpha, \beta$. This led Valeriote and Willard to pose the problem of discovering a tame congruence-theoretic characterization of locally finite congruence 3-permutable varieties. In this paper, we present such a characterization as well as a proof of the Valeriote–Willard Theorem which does not require the Idziak–McKenzie result. We finish the paper with an example which shows that, in a narrow sense, there is no "tame congruence-theoretic characterization" of congruence n-permutable locally finite varieties for $n \geqslant 4$.

Our reference for tame congruence theory is, of course, [4], while our reference for universal algebra is [7]. If α and β are binary relations on the same set, then we will use the notation $\alpha \circ_n \beta$ to denote the relation

$$\alpha \circ \beta \circ \alpha \circ \cdots$$

which is an iterated composition of α with β which begins with α and has $n-1$ occurrences of $\circ$. If α and β are equivalence relations, we say that α and β *n-permute* if $\alpha \vee \beta = \alpha \circ_n \beta = \beta \circ_n \alpha$. If $\mathbf{A}$ is an algebra, we say that $\mathbf{A}$ is *congruence n-permutable* if any pair of congruences on $\mathbf{A}$ n-permute. A variety $\mathscr{V}$ is congruence n-permutable if each member is. The statement that an algebra or a variety is congruence n-permutable may be written as a congruence equation or an inclusion:

$$\mathbf{A} \models_{\text{con}} \alpha \vee \beta \approx \alpha \circ_n \beta \qquad \text{or} \qquad \mathscr{V} \models_{\text{con}} \alpha \circ_n \beta \subseteq \beta \circ_n \alpha.$$

A *tolerance* on $\mathbf{A}$ is a reflexive, symmetric subalgebra of $\mathbf{A}\times\mathbf{A}$. We will express the fact that an algebra or a variety satisfies a tolerance inclusion or equation by writing, say,

$$\mathbf{A} \models_{\text{tol}} \tau \circ_n \tau \subseteq \tau \circ_{n-1} \tau \qquad \text{or} \qquad \mathcal{V} \models_{\text{tol}} \tau \circ_n \tau \approx \tau \circ_{n-1} \tau.$$

2. n-PERMUTABILITY

THEOREM 2.1. *For a variety $\mathcal{V}$ and an integer $n \geqslant 2$ the following conditions are equivalent*:

(i) *$\mathcal{V}$ is congruence n-permutable.*

(ii) *$\mathcal{V}$ has terms $p_i(x_0, ..., x_n)$, $i = 0, 1, ..., n$, such that*

$$\mathcal{V} \models p_0(\bar{x}) \approx x_0$$
$$p_i(x_0, x_2, x_2, x_4, x_4, ...) \approx p_{i+1}(x_0, x_2, x_2, x_4, x_4, ...) \qquad \textit{for } i \textit{ even}$$
$$p_i(x_1, x_1, x_3, x_3, x_5, ...) \approx p_{i+1}(x_1, x_1, x_3, x_3, x_5, ...) \qquad \textit{for } i \textit{ odd}$$
$$p_n(\bar{x}) \approx x_n.$$

(iii) $\mathcal{V} \models_{\text{tol}} \tau \circ_n \tau \subseteq \tau \circ_{n-1} \tau.$

Proof. The proof that (i) implies (ii) is a standard type of argument. We must show that if $\mathcal{V} \models_{\text{con}} \theta \circ_n \psi \subseteq \psi \circ_n \theta$, then $\mathcal{V}$ must have the terms described in (ii). On $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x_0, ..., x_n)$ let $\theta = \mathrm{Cg}^{\mathbf{F}}((x_0, x_1), (x_2, x_3), ...)$ and let $\psi = \mathrm{Cg}^{\mathbf{F}}((x_1, x_2), (x_3, x_4), ...)$. We have $(x_0, x_n) \in \theta \circ_n \psi \subseteq \psi \circ_n \theta$, so there are elements $x_0 = p_0, p_1, ..., x_n = p_n$ such that $(p_i, p_{i+1}) \in \psi$ for i even and $(p_i, p_{i+1}) \in \theta$ for i odd. If we let $p_i(\bar{x})$ be a term representing the element p_i, then this last conclusion is equivalent to the fact that the $p_i(\bar{x})$'s satisfy the equations in (ii).

Now, let's assume (ii) and prove that $\tau \circ_n \tau \subseteq \tau \circ_{n-1} \tau$. If $(a, b) \in \tau \circ_n \tau$, then we can find a sequence $x_0, x_1, ..., x_n$ such that $a = x_0$, $b = x_n$, and $(x_i, x_{i+1}) \in \tau$ for all $i < n$. Since τ is a tolerance, for $i = 1$ to $n-1$ we have

$$\begin{aligned} &p_i^{\mathbf{A}^2}((x_0, x_1), (x_2, x_1), (x_2, x_3), ..., (x_j, x_k), (x_n, x_n)) \\ &\quad = (p_i^{\mathbf{A}}(x_0, x_2, x_2, ..., x_j, x_n), p_i^{\mathbf{A}}(x_1, x_1, x_3, ..., x_k, x_n)) \\ &\quad = (y_i, z_i) \in \tau, \end{aligned}$$

where $(j, k) = (n-1, n)$ if n is odd and $(j, k) = (n, n-1)$ if n is even. The equations of (ii) imply that $a = y_1$, $z_i = z_{i+1}$ for $i < n-1$ odd, and $y_i = y_{i+1}$ for $i < n-1$ even, while $z_{n-1} = b$. Thus $a \tau z_1 = z_2 \tau y_2 = y_3 \tau z_3 = z_4 \tau \cdots \tau b$, implying that $(a, b) \in \tau \circ_{n-1} \tau$.

We must show that (iii) implies (i) to finish the proof. Assume (iii) and choose $\mathbf{A} \in \mathcal{V}$ with congruences $\alpha, \beta \in \operatorname{Con} \mathbf{A}$. The relation $\tau = (\alpha \circ \beta) \cap (\beta \circ \alpha)$ is a tolerance on $\mathbf{A}$. Clearly

$$\alpha \cup \beta \subseteq \tau \subseteq \alpha \vee \beta,$$

so, by (iii), $\tau \circ_{n-1} \tau$ is a congruence which is necessarily equal to $\alpha \vee \beta$; therefore

$$\alpha \vee \beta = \tau \circ_{n-1} \tau \subseteq (\alpha \circ \beta) \circ_{n-1} (\beta \circ \alpha) = \alpha \circ_n \beta \subseteq \alpha \vee \beta.$$

The congruence equation $\alpha \vee \beta = \alpha \circ_n \beta$ establishes (i). ∎

In Exercise 8.8 (1) of [4], one is asked to show that if $\mathbf{A}$ is a finite algebra in a congruence permutable variety, then $\mathbf{A}$ satisfies the condition that for all $\alpha \prec \beta$ in $\operatorname{Con} \mathbf{A}$ and for all $(a, b) \in \beta - \alpha$ there is a $u \equiv b \pmod{\alpha}$ with $\{a, u\} \subseteq N$ for some $\langle \alpha, \beta \rangle$-trace N. The authors of [8] dub this condition "the HM condition" and prove that a locally finite variety $\mathcal{V}$ is congruence permutable if and only if each finite $\mathbf{A} \in \mathcal{V}$ satisfies the HM condition and the condition that $\operatorname{typ}\{\mathbf{A}\} \cap \{1, 5\} = \varnothing$.

We will write $T_{\alpha,\beta}$ to denote the reflexive relation that is generated by the squares of all the $\langle \alpha, \beta \rangle$-traces:

$$T_{\alpha,\beta} = \{(x, x) \in A^2 \mid x \in A\} \cup \{N^2 \mid N \text{ is an } \langle \alpha, \beta \rangle\text{-trace}\}.$$

One can rewrite the HM condition as simply $\beta = T_{\alpha,\beta} \circ \alpha$ for all $\alpha \prec \beta$ in $\operatorname{Con} \mathbf{A}$. We define an analogous property satisfied by algebras in congruence n-permutable varieties for all $n \geqslant 2$.

Definition 2.2. If $\mathbf{A}$ is a finite algebra, then $\mathbf{A}$ satisfies the condition HM_n if $\beta = \rho \circ_{n-1} \rho$ whenever $\alpha \prec \beta$ in $\operatorname{Con} \mathbf{A}$ and $\rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta})$. A class of algebras satisfies the condition HM_n if all of its finite members do.

We are interested primarily in HM_2 and HM_3 in this paper. The condition HM_2 implies that when $\alpha \prec \beta$ we have

$$\beta = \rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta}) \subseteq T_{\alpha,\beta} \circ \alpha \subseteq \beta$$

or just $\beta = T_{\alpha,\beta} \circ \alpha$. Conversely, if $\beta = T_{\alpha,\beta} \circ \alpha$, then $\beta = \beta^{\cup} = (T_{\alpha,\beta} \circ \alpha)^{\cup} = \alpha \circ T_{\alpha,\beta}$ so

$$\beta = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta}) = \rho.$$

Thus, the HM condition is equivalent to the condition HM_2. The condition HM_3 implies that

$$\beta = \rho \circ \rho \subseteq (T_{\alpha,\beta} \circ \alpha) \circ (\alpha \circ T_{\alpha,\beta}) = T_{\alpha,\beta} \circ \alpha \circ T_{\alpha,\beta} \subseteq \beta$$

or that $\beta = T_{\alpha,\beta} \circ \alpha \circ T_{\alpha,\beta}$. This means that if $(x, y) \in \beta$ we can find $(u, v) \in \alpha$ such that $x = u$ or $\{x, u\}$ is contained in an $\langle \alpha, \beta \rangle$-trace and $v = y$ or $\{v, y\}$ is contained in an $\langle \alpha, \beta \rangle$-trace. This is all that we will need to know about HM_3 to prove our main theorem, Theorem 2.5.

THEOREM 2.3. *Every congruence n-permutable locally finite variety satisfies the condition* HM_n.

Proof. We will show that $\rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta})$ is a tolerance. Of course, ρ is reflexive and symmetric; we need to show that it is a subalgebra of $\mathbf{A}^2$. For this it will suffice to prove that $T_{\alpha,\beta} \circ \alpha$ is a subalgebra, for then $\rho = (T_{\alpha,\beta} \circ \alpha) \cap (T_{\alpha,\beta} \circ \alpha)^{\cup}$ is an intersection of subalgebras.

Choose $(a_i, b_i) \in T_{\alpha,\beta} \circ \alpha$, $i < k$, and t a k-ary term. We need to verify that $(t^{\mathbf{A}}(\bar{a}), t^{\mathbf{A}}(\bar{b})) \in T_{\alpha,\beta} \circ \alpha$. Find c_i such that for all $i < k$

$$a_i T_{\alpha,\beta} c_i \alpha b_i.$$

Pick an $\langle \alpha, \beta \rangle$-trace N and $(u, v) \in N^2 - \alpha$. From Theorem 9.14 of [4] we know that $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{2}, \mathbf{3}\}$. Consequently, $\mathbf{N} = \mathbf{A}|_N$ has a Mal'cev polynomial and so $N^2 = \mathrm{Cg}^{\mathbf{N}}(u, v) \circ \alpha|_N$ and

$$\mathrm{Cg}^{\mathbf{N}}(u, v) = \{(p(u), p(v)) \in N^2 \mid p \in \mathrm{Pol}_1\, \mathbf{A}|_N\}.$$

(See [7], Theorem 4.70(ii) for this last conclusion.) Any two $\langle \alpha, \beta \rangle$-traces are polynomially isomorphic, so

$$\begin{aligned} T_{\alpha,\beta} &= \{(x, x) \in A^2 \mid x \in A\} \cup \{N^2 \mid N \text{ is an } \langle \alpha, \beta \rangle\text{-trace}\} \\ &\subseteq \{(p(u), p(v)) \in A^2 \mid p \in \mathrm{Pol}_1\, \mathbf{A}\} \circ \alpha. \end{aligned}$$

This means that we may choose polynomials $p_i(x) \in \mathrm{Pol}_1\, \mathbf{A}$ such that $a_i = p_i(u)$ and $p_i(v)\, \alpha\, c_i$. Hence $t^{\mathbf{A}}(\bar{a}) = t^{\mathbf{A}}(\overline{p_i(u)}) = g(u)$ while $t^{\mathbf{A}}(\bar{c})\, \alpha\, t^{\mathbf{A}}(\overline{p_i(v)}) = g(v)$ where $g(x) = t^{\mathbf{A}}(\overline{p_i(x)}) \in \mathrm{Pol}_1\, \mathbf{A}$. If $(g(u), g(v)) \in \alpha$, then $t^{\mathbf{A}}(\bar{a})\, \alpha\, t^{\mathbf{A}}(\bar{b})$ in which case we are done. Otherwise, $g(N)$ is an $\langle \alpha, \beta \rangle$-trace and so $(g(u), g(v)) \in T_{\alpha,\beta}$ and

$$t^{\mathbf{A}}(\bar{a}) = g(u)\; T_{\alpha,\beta}\, g(v)\, \alpha\, t^{\mathbf{A}}(\bar{c})\, \alpha\, t^{\mathbf{A}}(\bar{b}).$$

In either case, $(t^{\mathbf{A}}(\bar{a}), t^{\mathbf{A}}(\bar{b})) \in T_{\alpha,\beta} \circ \alpha$.

Now that we have shown that $\rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta})$ is a tolerance, Theorem 2.1 proves that $\rho \circ_{n-1} \rho$ is a congruence. ρ is contained in β and properly contains α, so $\rho \circ_{n-1} \rho = \beta$ and the condition HM_n holds. ∎

Next we prove a generalization of Theorem 8.1 of [3] which will be used in a small way in the proof of our main result, Theorem 2.5. For the next theorem we will use the notation that $\alpha^{[0]} = \alpha$ and $\alpha^{[k+1]} = [\alpha^{[k]}, \alpha^{[k]}]$.

The proof uses Theorem 8.1 of [3] which states that when $\mathbf{A}$ generates a congruence modular variety and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, then α permutes with β if and only if $\alpha^{[k]}$ permutes with $\beta^{[l]}$ for some $k, l \geqslant 0$.

THEOREM 2.4. *If* $\mathbf{A}$ *generates a congruence modular variety and* α *and* β *are congruences on* $\mathbf{A}$, *then* $\alpha \vee \beta = \alpha \circ_n \beta$ *if and only if* $\alpha^{[l]} \vee \beta^{[m]} = \alpha^{[l]} \circ_n \beta^{[m]}$ *for some* $l, m \geqslant 0$.

Proof. We will prove only that $\alpha \vee \beta = \alpha \circ_n \beta$ if and only if $[\alpha, \alpha] \vee \beta = [\alpha, \alpha] \circ_n \beta$. From our argument it will be clear how to show that $\alpha \vee \beta = \alpha \circ_n \beta$ if and only if $\alpha \vee [\beta, \beta] = \alpha \circ_n [\beta, \beta]$ and then an obvious induction establishes the theorem. As Theorem 8.1 of [3] proves the result for $n = 2$ we will only consider $n > 2$.

The congruences $[\alpha, \alpha]$ and $[\alpha, \alpha] \vee \beta$ are comparable, so they permute. By Theorem 8.1 of [3], α permutes with $[\alpha, \alpha] \vee \beta$, so

$$\alpha \circ ([\alpha, \alpha] \vee \beta) = ([\alpha, \alpha] \vee \beta) \circ \alpha = \alpha \vee ([\alpha, \alpha] \vee \beta) = \alpha \vee \beta.$$

If $[\alpha, \alpha] \vee \beta = [\alpha, \alpha] \circ_n \beta$, then $\alpha \vee \beta = \alpha \circ ([\alpha, \alpha] \circ_n \beta) \subseteq \alpha \circ_n \beta$. Thus, one of the desired implications,

$$[\alpha, \alpha] \vee \beta = [\alpha, \alpha] \circ_n \beta \rightarrow \alpha \vee \beta = \alpha \circ_n \beta,$$

has been proved.

Now assume that $\alpha \vee \beta = \alpha \circ_n \beta$. $\mathbf{A}$ generates a congruence modular variety, so we can find a ternary term $t(x, y, z)$ such that for all $x, y \in A$ we have $t(x, y, y) = x$ and

$$t(x, x, y)[\mathrm{Cg}^{\mathbf{A}}(x, y), \mathrm{Cg}^{\mathbf{A}}(x, y)]\, y.$$

Hence, if $(x, y) \in \beta \circ \alpha$ is arbitrary and $x \beta u \alpha y$, then

$$x \alpha t(x, u, y)\, \beta t(u, u, y)[\alpha, \alpha]\, y,$$

which proves that $\beta \circ \alpha \subseteq \alpha \circ \beta \circ [\alpha, \alpha]$. If $n > 2$ is odd an easy induction argument shows that

$$\begin{aligned} \alpha \circ_n \beta &= \alpha \circ \beta \circ \cdots \circ \alpha \circ \beta \circ \alpha \circ (\beta \circ \alpha) \\ &\subseteq \alpha \circ \beta \circ \cdots \circ \alpha \circ \beta \circ \alpha \circ (\alpha \circ \beta \circ [\alpha, \alpha]) \\ &\subseteq \alpha \circ \beta \circ \cdots \circ \alpha \circ \beta \circ [\alpha, \alpha] \circ \beta \circ [\alpha, \alpha] \\ &\subseteq \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha]) \subseteq \alpha \circ_n \beta \end{aligned}$$

or just $\alpha \circ_n \beta = \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha])$. If $n > 2$ is even we find similarly that

$$\begin{aligned}\alpha \circ_n \beta &= \alpha \circ \beta \circ \cdots \circ \beta \circ \alpha \circ (\beta \circ \alpha) \circ \beta \\ &\subseteq \alpha \circ \beta \circ \cdots \circ \beta \circ \alpha \circ (\alpha \circ \beta \circ [\alpha, \alpha]) \circ \beta \\ &\subseteq \alpha \circ \beta \circ \cdots \circ \beta \circ [\alpha, \alpha] \circ \beta \circ [\alpha, \alpha] \circ \beta \\ &\subseteq \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha]) \subseteq \alpha \circ_n \beta\end{aligned}$$

or again $\alpha \circ_n \beta = \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha])$.

Now suppose that $\alpha \vee \beta = \alpha \circ_n \beta = \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha])$. Choose $(a, b) \in [\alpha, \alpha] \vee \beta \subseteq \alpha \vee \beta = \alpha \circ (\beta \circ_{n-1} [\alpha, \alpha])$. We can find y_i, $i \leqslant n$, such that

$$a = y_0 \alpha y_1 \beta y_2 [\alpha, \alpha]\, y_3 \cdots y_n = b.$$

Now, $(y_0, y_1) \in \alpha \wedge ([\alpha, \alpha] \vee \beta)$ which equals $[\alpha, \alpha] \vee (\alpha \wedge \beta)$ by modularity. But α and $\alpha \wedge \beta$ permute so, by Theorem 8.1 of [3], $[\alpha, \alpha]$ and $\alpha \wedge \beta$ permute. This gives us that $(y_0, y_1) \in [\alpha, \alpha] \circ (\alpha \wedge \beta) \subseteq [\alpha, \alpha] \circ \beta$. There is a $z \in A$ such that

$$y_0 [\alpha, \alpha]\, z \beta y_1$$

and so

$$a = y_0 [\alpha, \alpha]\, z \beta y_2 [\alpha, \alpha]\, y_3 \cdots y_n = b.$$

That is, $(a, b) \in [\alpha, \alpha] \circ_n \beta$. The pair $(a, b) \vee \beta$ was chosen arbitrarily, so $[\alpha, \alpha] \vee \beta \subseteq [\alpha, \alpha] \circ_n \beta$ which finishes the proof of the implication

$$\alpha \vee \beta = \alpha \circ_n \beta \rightarrow [\alpha, \alpha] \vee \beta = [\alpha, \alpha] \circ_n \beta. \quad \blacksquare$$

THEOREM 2.5. *For $n = 2, 3$, a locally finite variety is congruence n-permutable if and only if for each finite $\mathbf{A} \in \mathcal{V}$,*

(i) $\operatorname{typ}\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}\}$,

(ii) *for each $\alpha \prec \beta$ in* Con $\mathbf{A}$, *the $\langle \alpha, \beta \rangle$-minimal sets have empty tail, and*

(iii) $\mathbf{A}$ *satisfies the condition* HM_n.

Proof. We will write the proof for the value $n = 3$ and include parenthetical remarks to show where the argument is different when $n = 2$.

$\mathcal{V}$ is congruence n-permutable and also congruence modular since $n = 3$ ($n = 2$). It follows from Theorems 8.5 and 9.14 of [4] that (i) and (ii) hold. Theorem 2.3 shows that condition (iii) holds.

Conversely, suppose that $\mathcal{V}$ satisfies (i), (ii), and (iii). Again, Theorem 8.5 of [4] implies that $\mathcal{V}$ is congruence modular. In order to obtain a contradiction assume that $\mathscr{V}$ contains an algebra $\mathbf{A}$ which has congruences β and δ such that $\beta \vee \delta \neq \beta \circ_n \delta$. We can choose $\mathbf{A}$ to be finite and choose β minimal for a non-equality of the form $\beta \vee \delta \neq \beta \circ_n \delta$. Hence if α is a fixed lower cover of β, then we have $\alpha \vee \gamma = \alpha \circ_n \gamma$ for any $\gamma \in \operatorname{Con} \mathbf{A}$. Now we may assume that δ is maximal for $\beta \vee \delta \neq \beta \circ_n \delta$ for this fixed value of β. We cannot have $\beta \leqslant \delta$, since β and δ do not permute. The minimality of β implies that $\alpha \vee \delta = \alpha \circ_n \delta$ so, if $\alpha \nleqslant \delta$, the maximality of δ implies that

$$\beta \vee \delta = \beta \vee (\alpha \vee \delta) = \beta \circ (\alpha \circ \delta \circ \alpha) \circ \beta = \beta \circ \delta \circ \beta$$

$$(\beta \vee \delta = \beta \vee (\alpha \vee \delta) = \beta \circ (\alpha \circ \delta) = \beta \circ \delta)$$

which is false. Hence $\alpha \leqslant \delta$. Let $\theta = \beta \vee \delta$; by modularity, $\delta \prec \theta$.

Now, $\beta \vee \delta \neq \beta \circ_n \delta$ but $\alpha \vee \delta = \alpha \circ_n \delta$. We cannot have $\operatorname{typ}(\alpha, \beta) = \mathbf{2}$ by Theorem 2.4 applied in $\mathbf{A}/\alpha$. Hence $\mathbf{3} = \operatorname{typ}(\alpha, \beta) = \operatorname{typ}(\delta, \theta)$. The members of $\operatorname{Min}_{\mathbf{A}}(\alpha, \beta) = \operatorname{Min}_{\mathbf{A}}(\delta, \theta)$ are two-element sets, each constituting a trace. From this we get that $T_{\delta,\theta} = T_{\alpha,\beta} \subseteq \beta$. Condition (iii) implies that

$$\beta \vee \delta = \theta = T_{\delta,\theta} \circ \delta \circ T_{\delta,\theta} = T_{\alpha,\beta} \circ \delta \circ T_{,\beta} \subseteq \beta \circ \delta \circ \beta,$$

$$(\beta \vee \delta = \theta = T_{\delta,\theta} \circ \delta = T_{\alpha,\beta} \circ \delta \subseteq \beta \circ \delta),$$

which is contrary to our assumption. ∎

3. Remarks and Reformulations

The principal result of [8] is more than just a characterization of the locally finite congruence permutable varieties; it is a statement about individual algebras. The authors of [8] prove that any finite algebra with type-set contained in $\{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ is congruence permutable if it satisfies the HM condition. One can strengthen our result concerning congruence 3-permutability to a similar statement:

Theorem 3.1. *If* $\mathbf{A}$ *satisfies the condition* HM_3 *and* $\operatorname{type}\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$, *then* $\mathbf{A}$ *has* 3-*permuting congruences.*

An alternate local reformulation of Theorem 2.5 is the following.

Theorem 3.2. *For a finite algebra* $\mathbf{A}$ *and for* $n = 2, 3$ *if* $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$ *and* $\mathbf{HS}(\mathbf{A}^2)$ *satisfies the condition* HM_n, *then* $\mathbf{A}$ *has n-permuting congruences.*

The arguments for these results are a bit longer than our proof of Theorem 2.5 and are nearly identical to those supplied by Valeriote and

Willard in [8]. Since the results claimed here are stronger than Theorem 2.5 we will prove Theorems 3.1 and 3.2 in this section.

The essential difference between our approach to Theorem 3.1 and the proof of Valeriote and Willard in the 2-permutable case is that some substitute must be found for the Idziak–McKenzie Theorem. This is the theorem that states that a finite algebra with a pair of non-permuting congruences has a pair of non-permuting congruences satisfying $\alpha \wedge \beta \prec \alpha, \beta$. A suitable substitute is the following.

LEMMA 3.3. *If the finite algebra* **A** *has a pair of congruences which fail to* 3*-permute, then one can find a pair* β, δ *for which* $\beta \vee \delta \not\subseteq \beta \circ \delta \circ \beta$ *and which satisfy* $\beta \wedge \delta \prec \beta$ *and either* $\beta \wedge \delta \prec \delta$ *or* $\delta \prec \beta \vee \delta$.

Proof. Assume that **A** has a pair of congruences which fail to 3-permute but has no such pair of congruences where each covers their meet. We will show that **A** has a pair of congruences β and δ for which $\beta \vee \delta \not\subseteq \beta \circ \delta \circ \beta$, $\beta \wedge \delta \prec \beta$, and $\delta \prec \beta \vee \delta$.

First assume that **A** has congruences α, θ, and ψ where $\alpha \wedge \theta \prec \alpha, \theta$, but $\theta < \psi < \alpha \vee \theta$. Choose $(a, b) \in \psi - \theta$. Since $(a, b) \in \alpha \vee \theta = \theta \circ \alpha \circ \theta$ there are elements $u, v \in A$ such that $a \theta u \alpha v \theta b$. Hence, $u \theta a \psi b \theta v$ and so $(u, v) \in \psi \wedge \alpha \subseteq \theta$. This forces $(a, b) \in \theta$ which is false. It follows that the congruence lattice of **A** satisfies the implication

$$\alpha \wedge \theta \prec \alpha, \theta \rightarrow \alpha, \theta \prec \alpha \vee \theta.$$

This fact is precisely the statement that **A** has a *semimodular* congruence lattice. Induction on the height of the congruence lattice establishes the apparently stronger implication that

$$\alpha \wedge \theta \prec \alpha \rightarrow \theta \prec \alpha \vee \theta.$$

(The argument for this is well-known. It is a consequence of Theorem 3.7 of [1], for example.) Now we can mimic the argument in the third paragraph of Theorem 2.5 which shows that since **A** contains a pair of congruences that fail to 3-permute, there is a pair of congruences β and δ where $\beta \vee \delta \not\subseteq \beta \circ \delta \circ \beta$ and $\beta \wedge \delta \prec \beta$. By semimodularity, $\delta \prec \beta \vee \delta$. This is what we set out to prove. ∎

The next lemma is similar to Lemma 3.3 of [8].

LEMMA 3.4. *Let* **A** *be a finite algebra with congruences* β, *and* δ *where* $\delta \prec \beta \vee \delta = \theta$ *and* $\mathrm{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. *Suppose that the prime quotient* $\langle \delta, \theta \rangle$ *satisfies the condition* HM_3. *Then* $\beta \vee \delta = \beta \circ \delta \circ \beta$.

Proof. Choose an arbitrary pair $(a, b) \in \beta \vee \delta = \theta$ and then choose $u, v \in A$ such that $a T_{\delta,\theta} u \, \delta v T_{\delta,\theta} b$. If $a \neq u$ and $v \neq b$, then there are $\langle \delta, \theta \rangle$-

traces N and N' such that $\{a, u\} \subseteq N$ and $\{v, b\} \subseteq N'$. Assume for now that this is the case. Choose $\langle \delta, \theta \rangle$-minimal sets U and U' which contain N and N', respectively, and let B and B' be their respective bodies. Since the type of the quotient $\langle \delta, \theta \rangle$ is **2**, **3**, or **4** the algebras $\mathbf{A}|_B$ and $\mathbf{A}|_{B'}$ are congruence permutable.

Both a and u belong to N so $(a, u) \in \theta|_B$ and similarly $(v, b) \in \theta|_{B'}$. Hence $(a, b) \in (\theta|_B) \circ \delta \circ (\theta|_{B'})$. From Lemma 2.4 of [4] we know that since $\theta = \beta \vee \delta$ we have $\theta|_B = \beta|_B \vee \delta|_B$ in $\operatorname{Con} \mathbf{A}|_B$. As $\mathbf{A}|_B$ is congruence permutable, it follows that this join is equal to $\beta|_B \circ \delta|_B$. A similar argument shows that $\theta|_{B'} = \delta|_{B'} \circ \beta|_{B'}$. Hence, $(a, u) \in \beta|_B \circ \delta|_B \subseteq \beta \circ \delta$ and $(v, b) \in \delta|_{B'} \circ \beta|_{B'} \subseteq \delta \circ \beta$. This conclusion is valid even in the cases when $a = u$ or $v = b$. We conclude that in all cases $(a, b) \in (\beta \circ \delta) \circ \delta \circ (\delta \circ \beta) = \beta \circ \delta \circ \beta$ as required. ∎

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By combining Lemmas 3.3 and 3.4 we find that it suffices to prove that if $\operatorname{typ}\{\mathbf{A}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and $\mathbf{A}$ satisfies the condition HM_3, then congruences α and β in $\operatorname{Con} \mathbf{A}$ which cover their meet must 3-permute. (The reason for this is that the other possibility mentioned in Lemma 3.3 is ruled out by Lemma 3.4.)

Let α and β be a pair of congruences on $\mathbf{A}$ where each covers their meet and let γ and θ be their meet and join, respectively. Choose congruences α' and β' such that $\alpha \leqslant \alpha' \prec \theta$ and $\beta \leqslant \beta' \prec \theta$. If $\alpha' \wedge \beta' \neq \gamma$, then the congruences α, β, and $\alpha' \wedge \beta'$ generate a sublattice of $\operatorname{Con} \mathbf{A}$ isomorphic to the lattice $\mathbf{D}_1$ as pictured in Fig. 1. This contradicts Lemma 6.4 of [4] since $\mathbf{A}$ omits type **1**. Thus $\alpha' \wedge \beta' = \gamma$. By Lemma 3.4, using the congruences α and β' with $\beta' \prec \alpha \vee \beta' = \theta$ we conclude that $\theta = \alpha \circ \beta' \circ \alpha$. Using the congruences β and α' with $\alpha' \prec \theta$ we get $\theta = \beta \circ \alpha' \circ \beta$. We will finish the proof by showing that the previously established facts $\theta = \alpha \vee \beta = \alpha \circ \beta' \circ \alpha = \beta \circ \alpha' \circ \beta$, $\alpha \leqslant \alpha'$, $\beta \leqslant \beta'$, and $\alpha' \wedge \beta' = \alpha \wedge \beta$ imply that $\theta = \alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$.

Choose an arbitrary pair $(a, b) \in \theta$ and then choose $u, v \in A$ such that $a \alpha u \beta' v \alpha b$. Since $\beta' \leqslant \theta = \beta \circ \alpha' \circ \beta$, we can find $p, q \in A$ such that $u \beta p \alpha' q \beta v$.

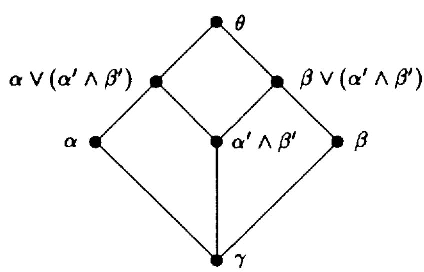

FIGURE 1

But now, $(p, q) \in \alpha' \cap (\beta \circ \beta' \circ \beta) \subseteq \alpha' \wedge \beta' \subseteq \beta$. Hence $(u, v) \in \beta$ and so $(a, b) \in \alpha \circ \beta \circ \alpha$. The pair $(a, b) \in \theta$ was arbitrary, so $\theta = \alpha \circ \beta \circ \alpha$. The facts we used from the previous paragraph came from a list of facts symmetric in α and β, so $\theta = \beta \circ \alpha \circ \beta$, too. ∎

It turns out that all finite distributive lattices and all finite semilattices satisfy the condition HM_4, but not all satisfy HM_3 or HM_2. These failures can be interpreted into any locally finite variety which admits the lattice or semilattice type by using the technique described in Theorem 6.17 of [4].

LEMMA 3.5. *If* $\mathbf{A}$ *is a finite algebra and* $\mathbf{HS}(\mathbf{A}^2)$ *satisfies the condition* HM_n *for* $n = 2$ *or* 3, *then* $\mathrm{typ}\{\mathbf{A}\} \subseteq \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. *If* $\mathbf{HS}(\mathbf{A}^3)$ *satisfies the condition* HM_n *for* $n = 2$ *or* 3, *then* $\mathrm{typ}\{\mathbf{A}\} \subseteq \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$.

Proof. We will only prove the first statement. The second can be proved in a similar fashion and in any case is not needed anywhere in this paper.

Assume that $\mathbf{A}$ has a prime congruence quotient of type $\mathbf{5}$. Then some homomorphic image of $\mathbf{A}$, call it $\mathbf{B}$, has a minimal congruence α where $\mathrm{typ}(0, \alpha) = \mathbf{5}$. We will show that some subalgebra of $\mathbf{B}^2$ fails the condition HM_3. Since HM_3 is weaker than HM_2 and $\mathbf{S}(\mathbf{B}^2) \subseteq \mathbf{HS}(\mathbf{A}^2)$, this will finish the proof.

Let U be a $\langle 0_B, \alpha \rangle$-minimal set with body equal to $N = \{0, 1\}$ and let $e \in E(\mathbf{B})$ be an idempotent unary polynomial for which $e(B) = U$. If Δ denotes the set of diagonal elements of $\mathbf{B}^2$, define $\mathbf{R} = \mathrm{Sg}^{\mathbf{B}^2}(N^2 \cup \Delta)$. We will show that $\mathbf{R}$ fails HM_3.

The algebra $\mathbf{R}$ is closely examined in Theorem 5.27 of [4] and we will quote facts about $\mathbf{R}$ directly from the proof of that theorem. Let $a = (1, 1)$, $b = (0, 1)$, $c = (1, 0)$, and $d = (0, 0)$. Let $\beta = \mathrm{Cg}^{\mathbf{R}}(c, d)$ and $\delta = \mathrm{Cg}^{\mathbf{R}}((a, c), (b, d))$. Let $e'(x, y) = (e(x), e(y))$ be the idempotent unary polynomial of $\mathbf{R}$ which is just e acting coordinatewise and let $U' = e'(R)$. By Theorem 5.27 of [4] we have $0_R \prec \beta$ and $\delta \prec \beta \vee \delta = \theta$ and $(a, b) \in \theta - \delta$. To show that HM_3 fails in $\mathbf{R}$ we will prove that $(a, b) \in \theta - (T_{\delta,\theta} \circ \delta \circ T_{\delta,\theta})$.

Our argument will require certain facts about $\mathbf{R}|_{N^2}$ which we collect here. $N^2 = \{a, b, c, d\}$ is a $\theta|_{U'}$-class since

$$N^2 \subseteq a/\theta|_{U'} \subseteq (1, 1)/(\alpha \times \alpha)|_{e'(R)} \subseteq (1/(\alpha|_{e(B)})) \times (1/(\alpha|_{e(B)})) = N^2.$$

$\mathbf{R}|_{N^2}$ is polynomially equivalent to the square of the 2-element semilattice by the last paragraph of Theorem 5.27 of [4]. The only non-trivial $\beta|_{N^2}$-class is $\{c, d\}$; the non-trivial $\delta|_{N^2}$-classes are $\{a, c\}$ and $\{b, d\}$. This follows from the description of the congruences on $\mathbf{R}|_{N^2}$ given in the claim of Theorem 5.27. Finally, $(T_{\delta,\theta})|_{N^2} = T_{\delta|_{N^2}, \theta|_{N^2}}$. This follows from the definition of $\mathbf{R}|_{N^2}$ and of $T_{\delta,\theta}$.

Since $\mathbf{R}|_{N^2}$ is polynomially equivalent to a 2-element semilattice, it is a matter of straightforward computation to verify that

$$T_{\delta|_{N^2},\,\theta|_{N^2}} = \{(c, d), (d, c), (a, a), (b, b), (c, c), (d, d)\}.$$

A consequence of this and the facts of the last paragraph is that (a, x), $(b, y) \in (T_{\delta,\theta})|_{U'}$ implies that $x = a$ and $y = b$. We are now in position to conclude our proof that $(a, b) \in \theta - (T_{\delta,\theta} \circ \delta \circ T_{\delta,\theta})$.

Clearly we have $(a, b) \in \delta \circ \beta \circ \delta \subseteq \theta$. Assume that $aT_{\delta,\theta}u\ \delta v T_{\delta,\theta} b$. Then

$$a = e'(a)\, e'(T_{\delta,\theta})\, e'(u)\ \delta e'(v)\, e'(T_{\delta,\theta})\, e'(b) = b.$$

The definition of $T_{\delta,\theta}$ implies that $e'(T_{\delta,\theta}) \subseteq \delta|_{U'} \cup T_{\delta,\theta}|_{U'}$. Since $(a, e'(u)) \in e'(T_{\delta,\theta}) \subseteq T_{\delta,\theta}|_{U'} \cup \delta$ we get either $(a, e'(u)) \in \delta$ or, by the last paragraph, $a = e'(u)$. Both cases tell us that $(a, e'(u)) \in \delta$ and a similar argument shows that $(e'(v), b) \in \delta$. Hence $(a, b) \in \delta \circ \delta \circ \delta = \delta$ which is false. This contradiction shows that the condition HM_3 fails in $\mathbf{R}$. ▌

Theorem 3.2 is a corollary of Lemma 3.5 and either Theorem 3.1 in the 3-permutable case or Theorem 3.4 of [8] in the 2-permutable case. From Theorem 3.2 we obtain stronger versions of Theorem 2.5.

COROLLARY 3.6. *For $n = 2, 3$, a locally finite variety is congruence n-permutable if and only if for each finite $\mathbf{A} \in \mathscr{V}$,*

(i) $\mathbf{1} \notin \operatorname{typ}\{\mathbf{A}\}$ *and*

(ii) $\mathbf{A}$ *satisfies the condition* HM_n.

It is still true that in a congruence 2- or 3- permutable variety all the $\langle \alpha, \beta \rangle$-minimal sets have empty tail as is stated in Theorem 2.5(ii). Conversely, this condition holding throughout a variety $\mathscr{V}$ implies that $\mathbf{1} \notin \operatorname{typ}\{\mathscr{V}\}$. (This is proved in Theorem 5.4 of [6].)

COROLLARY 3.7. *For $n = 2, 3$, a locally finite variety is congruence n-permutable if and only if for each finite $\mathbf{A} \in \mathscr{V}$,*

(i) *for each $\alpha \prec \beta$ in* $\operatorname{Con} \mathbf{A}$, *the $\langle \alpha, \beta \rangle$-minimal sets have empty tail, and*

(ii) $\mathbf{A}$ *satisfies the condition* HM_n. ▌

None of the sets of conditions described in this paper characterize congruence n-permutable locally finite varieties for $n \geqslant 4$. We have seen that for any fixed n a congruence n-permutable variety must have a type-set contained in $\{\mathbf{2}, \mathbf{3}\}$ and must satisfy the condition HM_n. These are the only tame congruence-theoretic properties of congruence n-permutable varieties that we know. However, these conditions are not enough to characterize congruence n-permutability for any $n \geqslant 4$ as the next example shows.

EXAMPLE 3.8. Let $\mathbf{L}$ be the nth power of the two-element lattice and let C denote the set of elements in some fixed maximal chain in $\mathbf{L}$. Let F be the set of all operations on L which are compatible with the lattice congruences of $\mathbf{L}$ and which preserve C. Let $\mathbf{A}$ be the algebra $\langle L; F\rangle$ and let $\mathbf{C}$ be the subalgebra of $\mathbf{A}$ whose universe is C.

First, let's show that for $\mathscr{V} = \mathscr{V}(\mathbf{A})$ we have $\operatorname{typ}\{\mathscr{V}\} \subseteq \{\mathbf{2}, \mathbf{3}\}$. Since F contains the lattice operations, $\mathscr{V}$ is congruence distributive. $\mathbf{A}$ has the same set of congruences as $\mathbf{L}$, so $\mathbf{A}$ is a product of n two-element algebras. It follows from congruence distributivity that $\mathscr{V}$ is residually $\leqslant 2$. One can check that each of these two-element algebras is primal, so they each have type-set equal to $\{\mathbf{3}\}$; thus, $\operatorname{typ}\{\mathscr{V}\} = \{\mathbf{3}\} \subseteq \{\mathbf{2}, \mathbf{3}\}$.

Now we show that $\mathscr{V}$ satisfies the condition HM_4. Choose a finite algebra $\mathbf{D} \in \mathscr{V}$ and congruences $\alpha \prec \beta$ in $\operatorname{Con} \mathbf{D}$. The proof of Theorem 10.16 of [2] shows that, since $\mathscr{V}$ is generated by two-element algebras, each β-class contains at most two different α-classes. Any $\langle \alpha, \beta\rangle$-trace must connect two different α-classes in some β-class and any β-class is connected, modulo α, by the $\langle \alpha, \beta\rangle$-traces that it contains. From this it follows that $\alpha \circ T_{\alpha,\beta} \circ \alpha = \beta$. If $\rho = (T_{\alpha,\beta} \circ \alpha) \cap (\alpha \circ T_{\alpha,\beta})$, then $\alpha \cup T_{\alpha,\beta} \subseteq \rho$, so

$$\beta = \alpha \circ T_{\alpha,\beta} \circ \alpha \subseteq \rho \circ \rho \circ \rho \subseteq \beta.$$

Since β is equal to $\rho \circ \rho \circ \rho$ whenever $\alpha \prec \beta$ in $\operatorname{Con} \mathbf{D}$ and $\mathbf{D}$ was chosen arbitrarily, $\mathscr{V}$ satisfies HM_4.

Finally, we must show that $\mathscr{V}$ is not congruence n-permutable. To do this we will exhibit a pair of congruences on $\mathbf{C}$ which do not n-permute. Write $C = \{x_0, ..., x_n\}$ for the universe of $\mathbf{C}$ where the elements are numbered so that $x_0 \prec \cdots \prec x_n$ in $\mathbf{L}$. The equivalence relations, θ, ψ, on C generated respectively by $\{(x_0, x_1), (x_2, x_3), ...\}$ and $\{(x_1, x_2), (x_3, x_4), ...\}$ are congruences on $\mathbf{C}$ for which $\theta \circ_n \psi \neq \psi \circ_n \theta$.

The variety described in Example 3.8 is a finitely generated, congruence distributive variety of type-set $\{\mathbf{3}\}$ which is residually $\leqslant 2$. This variety is essentially of finite type, since $\mathbf{Clo\ A}$ contains a 3-ary near unanimity operation derived from the lattice operations on $\mathbf{A}$. Thus, even in the nicest possible situation we find that the condition HM_n fails to characterize congruence n-permutability for any fixed $n \geqslant 4$. This leads us to ask the following.

Question 1. For which n is there a tame congruence-theoretic characterization of congruence n-permutability for locally finite varieties?

This question may be unanswerable since it is not clear what is meant by a "tame congruence-theoretic characterization." Tame congruence theory is a study of induced algebras corresponding to pairs of congruences $\alpha < \beta$ on a finite algebra where the interval $I[\alpha, \beta]$ is tame. The viewpoint of this paper is that congruence n-permutability is a statement about tolerances. Therefore, a narrow reframing of Question 1 is the following.

Question 2. For which n is the following state true? Let $\mathscr{V}$ be a locally finite variety whose type-set is contained in $\{\mathbf{2}, \mathbf{3}\}$. The condition $\mathbf{A} \models_{\text{tol}} \tau \circ_n \tau \approx \tau \circ_{n-1} \tau$ for all finite $\mathbf{A} \in \mathscr{V}$ and for all tolerances τ holds if and only if it holds for those τ satisfying $\alpha \subseteq \tau \subseteq \beta$ whenever $I[\alpha, \beta]$ is a tame interval in $\operatorname{Con} \mathbf{A}$.

Question 2 asks for those n for which the tolerance equation $\mathscr{V} \models_{\text{tol}} \tau \circ_n \tau \approx \tau \circ_{n-1} \tau$ can be deduced from its instances where τ is "trapped" inside a tame interval of $\operatorname{Con} \mathbf{A}$ for $\mathbf{A}$ finite. The answer to Question 2 is "only for $n = 2$ or 3." The statement in Question 2 is true when $n = 2$ or 3 by Corollary 3.6. On the other hand, Example 3.8 provides a counterexample to the statement in Question 2 for each $n \geqslant 4$. So, if we adopt a sufficiently narrow interpretation of the phrase, we find that there is no "tame congruence-theoretic characterization" of congruence n-permutable locally finite varieties for $n \geqslant 4$.

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