Strong Induction,
Course-of-values Recursion

Theorem.

Strong Induction

Theorem. (Page 44)

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula.

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof.

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$.

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$. Applying ordinary induction to $\theta(x)$ has the effect of applying strong induction to $\varphi(x)$.

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$. Applying ordinary induction to $\theta(x)$ has the effect of applying strong induction to $\varphi(x)$. $\square$

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$. Applying ordinary induction to $\theta(x)$ has the effect of applying strong induction to $\varphi(x)$. $\square$

Example.

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$. Applying ordinary induction to $\theta(x)$ has the effect of applying strong induction to $\varphi(x)$. $\square$

Example.
Define the Fibonacci sequence by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Strong Induction

Theorem. (Page 44) Assume that $\varphi(x)$ is a formula. If

1. $\varphi(0)$ is true and
2. $((\forall k \leq n)\varphi(k))$ implies $\varphi(S(n))$,

then $\varphi(n)$ is true for all n.

Idea of proof. Let $\theta(x)$ be "$((\forall k \leq n)\varphi(k))$. Applying ordinary induction to $\theta(x)$ has the effect of applying strong induction to $\varphi(x)$. □

Example.
Define the Fibonacci sequence by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.
Show that

$$\left(\frac{3}{2}\right)^n \leq F_{n+1} \leq 2^n.$$

Course-Of-Values Recursion

Course-Of-Values Recursion

Theorem.

Course-Of-Values Recursion

Theorem. (Page 50)

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E\colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E\colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E\colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.
3. Use E to define F.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E\colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.
3. Use E to define F.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G \colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F \colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E \colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.
3. Use E to define F.

Example.

Course-Of-Values Recursion

Theorem. (Page 50) Let A be a set, $a \in A$, and let A^* be the set of finite sequences of elements of A. If $G\colon A^* \times \mathbb{N} \to A$ is a function, then there is a unique function $F\colon \mathbb{N} \to A$ satisfying

1. $F(0) = a$.
2. $F(S(n)) = G(F|_{S(n)}, n)$. (Here $F|_{S(n)}$ is considered to be the sequence $(F(0), F(1), \ldots, F(n))$.)

Ideas of proof.

1. Show that is A is a set, then A^* is a set.
2. Apply ordinary recursion to define the function $E\colon \mathbb{N} \to A^*$ where $E(n) = (F(0), F(1), \ldots, F(n))$.
3. Use E to define F.

Example.
Define the Fibonacci sequence by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.