SET THEORY
MIDTERM

You have 50 minutes for this exam. You may not use any unauthorized sources, and you may not communicate with others about the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

I have neither given nor received aid on this exam.

Name:______________________________

1. Write the Axiom of Extensionality in the formal language of set theory. (There should be no English in your answer. Use $\in, =, \forall, \exists, \wedge, \vee, \neg, \rightarrow, \leftrightarrow, x_0, x_1, x_2, \ldots$.)

$$(\forall x_0)(\forall x_1)((x_0 = x_1) \longleftrightarrow (\forall x_2)((x_2 \in x_0) \longleftrightarrow (x_2 \in x_1)))$$

2.

(a) Define *"coimage of a function"* and give an example.

The coimage of a function $f \colon A \to B$ is the set of fibers $\mathrm{coim}(f) = \{f^{-1}(b) \mid b \in \mathrm{im}(f)\}$.

Example: If $f = id_A \colon A \to A$ is the identity function on $A = \{0, 1\}$, then $\mathrm{coim}(f) = \{\{0\}, \{1\}\}$.

(b) Define *"well-ordered set"* and give an example.

An ordered set $(X; <)$ is *well-ordered* if every nonempty subset of X has a least element.

Example: $(\mathbb{N}; <)$.

3. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ implies $A \subseteq B$.

Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$, we derive that $A \in \mathcal{P}(B)$, hence $A \subseteq B$.

4. Let T be a transitive set with exactly one element. What are the possibilities for T?

The only possibility is $T = \{\emptyset\}$.

To see this, assume to the contrary that $T = \{x\}$ is transitive and $x \neq \emptyset$. For any $y \in x$ we have $y \in x \in T$, so $y \in T = \{x\}$, so $y = x$. This leads to $x = y \in x$, or $x \in x$, which contradicts the Axiom of Foundation.

5. Show that $m + n = 0$ implies $m = n = 0$ for any $m, n \in \mathbb{N}$.

Let $\varphi(x)$ be the formula $(\forall m)((m + x = 0) \to ((m = 0) \wedge (x = 0))$. We argue by induction that the natural numbers satisfies $(\forall n)\varphi(n)$. To apply the Principle of Induction, we need to establish that $\varphi(0)$ holds and that $\varphi(n) \Rightarrow \varphi(S(n))$ holds for every natural number n.

(Base case: $\varphi(0)$ holds.)

Assume that $m + 0 = 0$. Since $m + 0 = m$ (according to the recursive definition of $+$), we get $m = m + 0 = 0$, so $m = 0$ $(= n)$.

(Inductive step. $\varphi(n)$ implies $\varphi(S(n))$)

To prove the implication $(m + S(n) = 0) \to ((m = 0) \wedge (S(n) = 0))$, start by assuming that $m + S(n) = 0$. By the recursive definition of $+$, $S(m + n) = 0$. But 0 is not a successor, so we have a contradiction. This shows that the premise of the implication $(m + S(n) = 0) \to ((m = 0) \wedge (S(n) = 0))$ is false for any m and n, so the implication is true for any m and n.