SET THEORY

SUMMARY OF TOPICS FROM 10/4/23-12/13/23

IV. Cardinality of sets
  (a) Equipotence ($|A| = |B|$). $|A| \leq |B|$, $|A| \leq m$, etc.
  (b) Cantor-Bernstein-Schröder Theorem.
  (c) Finite sets and their properties.
  (d) Countable sets.

V. Ordinals and cardinals.
  (a) Transitive sets.
  (b) Definition of ordinal and cardinal number.
  (c) Equipotence classes of infinite ordinals are bounded intervals.

VI. Ordinals.
  (a) Well-ordered sets. (Definition. Uniqueness properties from Section 6.1.3.)
  (b) Ordinal numbers. (Definition. Examples. Successor vs. limit ordinals. Ordinals are well-ordered. Order type. Natural numbers are exactly the finite ordinals.)
  (c) Axiom of Replacement.
  (d) The class of ordinal numbers is not a set.
  (e) Paradoxes of set theory: Russell's Paradox, Cantor's Paradox, Burali-Forti Paradox.
  (f) Transfinite induction.
  (g) Recursion Theorem.
  (h) Ordinal arithmetic. (Definitions, properties, order-theoretic interpretations.)

VII. Cardinals.
  (a) Definition (= initial ordinal) and notation.
  (b) Hartogs number of a set.
  (c) Definitions for cardinal arithmetic.
  (d) $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$.

VIII. Axiom of Choice.
  (a) ZF proves that A can be well-ordered iff $\mathcal{P}(A) \setminus \{\emptyset\}$ has a choice function.
  (b) Within ZF, AC is equivalent to:
    (i) Well-Ordering Theorem.
    (ii) Zorn's Lemma.
    (iii) Every surjective function has a right inverse.
    (iv) Every set is equipotent with a unique initial ordinal.
    (v) Any two sets have comparable cardinalities.
    (vi) $|A \times A| = |A|$ for any infinite A.
  (c) AC implies that
    (i) every infinite set has a countably infinite subset.

(ii) every Dedekind finite set is finite.
(iii) a countable union of countable sets is countable.
(iv) $2^{\aleph_0} \geq \aleph_1$.

IX. Cardinal Arithmetic.
(a) Cofinality.
(b) Every infinite cardinal λ can be partitioned into λ-many cofinal λ-sequences.
(c) Kőnig's Theorem (+ the corollary $\kappa^{\mathrm{cf}(\kappa)} > \kappa$).
(d) Regular and singular cardinals.
(e) Successor cardinals are regular. $\aleph_\omega$ and $\beth_\omega$ are singular.
(f) Within ZFC a cardinal κ may equal $|\mathbb{R}| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$ iff $\mathrm{cf}(\kappa)$ is uncountable.
(g) Cardinal exponentiation.
(h) Statements of the CH and the GCH.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Suppose I have a set of pairwise disjoint circles in $\mathbb{R}^2$, all of radius 1. Explain why my set is at most countable.

(2) Show that the intersection of two transitive sets is transitive. (What about union?)

(3) (a) Define "well-ordered set".
(b) Define "ordinal".
(c) Show that 0 is an ordinal and that the successor of an ordinal is an ordinal.
(d) Show that an element of an ordinal is an ordinal.

(4) Show that any well-ordered set that is embeddable in $\langle \mathbb{R}; < \rangle$ is embeddable in $\langle \mathbb{Q}; < \rangle$. (Here 'embedding' means 'order-preserving embedding'.) Conclude that a well-ordered sub-poset of the real numbers must be countable.

(5) Let P be the set of partial orderings of ω, let L be the set of all linear orderings of ω, and let W be the set of all well-orderings of ω. How are the cardinalities of the sets P, L, W related to the cardinalities of ω and $\mathcal{P}(\omega)$?

(6) What are

(a) Russell's Paradox?
(b) Cantor's Paradox?
(c) The Burali-Forti Paradox?

(7) Explain why the class of all ordinals is not a set.

(8) Is the set of proper subsets of ω, ordered by inclusion, an inductively ordered set? What conclusions, if any, can be drawn?

(9) If the poset $\langle P; \leq \rangle$ is inductively ordered, and $X \subseteq P$, must the subposet $\langle X; \leq \rangle$ be inductively ordered?

(10) Draw a picture of a poset that is not inductively ordered, but every element of the poset lies below some maximal element of the poset.

(11) Let X be a topological space and suppose that $x \in X$. Let $\mathcal{K}$ be the poset of open subsets of X that exclude x and let $\mathcal{L}$ be the poset of closed subsets of X that exclude x (both ordered by inclusion). Is either $\mathcal{K}$ or $\mathcal{L}$ inductively ordered?

(12) Explain why the Hartogs number of a set is a cardinal. What is the Hartogs number of $\aleph_\alpha$? What are the possibilities for the Hartogs number of an infinite, Dedekind finite set?

(13) Explain why the cofinality of an ordinal is a cardinal.

(14) Does it require the Axiom of Choice to prove that $|A \sqcup A| = |A|$ for infinite sets A?

(15) Show that if κ is an infinite cardinal, then $2^\kappa = \kappa^\kappa$.

(16) Show that $\mathrm{cf}\,(2^\kappa) > \kappa$.

(17) Explain why $|\mathbb{R}| \neq \aleph_\omega$.