

## SET THEORY

### SUMMARY OF TOPICS FROM 8/28/23-10/4/23

#### I. Axiomatic Set Theory

- (a) Extensionality.
- (b) Valid constructions of new sets (pairing, union, power set, comprehension, replacement, choice).
- (c) Empty set. Successor of a set. Successor is an injective class function. 0 is not a successor.
- (d) The Axiom of Infinity.
- (e) Intersection versus Union.
- (f) The directed graph models of set theory.
- (g) Naive set theory is inconsistent (Russell's Paradox).

#### II. Relations and Functions

- (a) Ordered pairs. Kuratowski encoding. Cartesian products.
- (b) Relations.
- (c) Functions, equivalence relations, partitions, kernel, image, coimage.
- (d) Ordered sets (strict vs. nonstrict, partial order vs. total order). Well-ordered set.

#### III. Natural numbers

- (a) Definition of  $\mathbb{N} = \omega$ .
- (b) Principle of Induction.
- (c) Recursion Theorem.
- (d) Transitive sets.  $\omega$  is well-ordered.
- (e) Arithmetic of  $\omega$ .

### General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

**Sample Problems.**

- (1) Why is naive set theory inconsistent?
- (2) Write the following axioms of set theory formally: Empty Set, Extensionality, and Union.
- (3) Explain why the intersection of two sets is a set.
- (4) Prove or disprove:
  - (a)  $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ .
  - (b)  $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ .
- (5) Define: Cartesian product, inductive set, function, partial order.
- (6) What is the kernel of the successor function? What is the image of the successor function? What is the coimage of the successor function? (The answers are classes.)
  
- (7) Explain why induction is a valid form of proof.
- (8) Show that  $m + n = 0$  implies  $m = n = 0$  for natural numbers  $m$  and  $n$ . Then show that  $m + n = 1$  implies that  $m$  and  $n$  are 0 and 1 in some order.
- (9) Show that the intersection of two transitive sets is transitive. (What about union?)
- (10) Show that  $T$  is transitive if and only if  $T \subseteq \mathcal{P}(T)$ .