Set Theory
Quiz 6

Name:______________________________

You have 10 minutes to complete this quiz. If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show your work, and give adequate explanations.

1. Let $A \cup B$ denote the union of A and B and $A \sqcup B$ denote the disjoint union of A and B. Show that $|A \cup B| \leq |A \sqcup B|$.

$A \cup B$ is the disjoint union of A and $B - A$ $(\subseteq B)$. We define an injection from the disjoint union $A \sqcup (B - A) = A \cup B$ to the disjoint union $A \sqcup B$ summand by summand: Let $\mathrm{id}_A \colon A \to A$ be the identity function on A (which injective) and let $j \colon (B - A) \to B$ be the inclusion function (which injective). Since $\mathrm{dom}(\mathrm{id}_A)$ is disjoint from $\mathrm{dom}(j)$ and $\mathrm{cod}(\mathrm{id}_A)$ is disjoint from $\mathrm{cod}(j)$, the union $\mathrm{id}_A \cup j = \mathrm{id}_A \sqcup j$ is an injective function from $A \sqcup (B - A) = A \cup B$ to $A \sqcup B$.

2. Show that a subset of a Dedekind finite set is Dedekind finite.

Solution 1. We prove the contrapositive: a superset of a Dedekind infinite set is Dedekind infinite.

We showed that D is Dedekind infinite iff $|\mathbb{N}| \leq |D|$. If $S \supseteq D$ is a superset, then $|\mathbb{N}| \leq |D| \leq |S|$, so $|\mathbb{N}| \leq |S|$ and S is Dedekind infinite.

Solution 2. We argue by contradiction. Assume that S is Dedekind finite and that some subset $D \subseteq S$ is Dedekind infinite. There must be a nonsurjective injection $f \colon D \to D$. Extend f to a nonsurjective injection $\widehat{f} \colon S \to S$ by

$$\widehat{f}(x) = \begin{cases} f(x) & x \in D \\ x & \text{else.} \end{cases}$$

Note that $\widehat{f}$ is nonsurjective, since $S - \widehat{f}(S) = D - f(D) \neq \emptyset$. But $\widehat{f}$ is injective, since $\widehat{f}(D) \subseteq D$ and $\widehat{f}|_D = f$ is injective and $\widehat{f}(S - D) \subseteq (S - D)$ and $\widehat{f}|_{S-D} = \mathrm{id}_{S-D}$ is injective.