1. (Exercise 8.1.9) Prove (using AC): Every infinite set is equipotent to some of its proper subsets. Equivalently, Dedekind finite sets are precisely the finite sets.

This was proved on Slide 6 of the slides from the October 23 lecture, but I will include a brief argument here.

We must explain why, under ZFC, Dedekind finite sets are finite. Contrapositively, we must explain why an infinite set A is Dedekind infinite. Using the Well-ordering Principle, enumerate A by some ordinal α. Since ω is infinite and its elements are finite, ω is the least infinite ordinal. Therefore, $\omega \subseteq \alpha$, so $|\omega| \leq |\alpha| = |A|$, so A is Dedekind infinite. (Here I am using a ZF-theorem that we proved: a set D is Dedekind infinite iff $|\omega| \leq |D|$.)

2. (Exercise 8.1.16.) Prove: If, for any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$, then the Axiom of Choice holds.

It suffices to argue that if, for any sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$, then every set can be well-ordered.

For this, choose A arbitrarily and let $B = h(A)$ (= the Hartogs number of A). Then $|h(A)| \leq |A|$ is impossible, so $|A| \leq |h(A)|$. This yields an injective function $f\colon A \to h(A)$. Well-order A by pulling back the order on the ordinal $h(A)$ to A: $x < y$ in A if $f(x) < f(y)$ in $h(A)$.

3. Suppose you are working in a universe of sets satisfying ZF in which all proper classes have the same size in the following sense: whenever C and D are proper classes, then there is a class bijection $F\colon C \to D$. Show that the Axiom of Choice holds in your universe.

Assume that $F\colon V \to \mathrm{ON}$ is a class bijection from the class of all sets to the class of ordinal numbers. We can use F to define a choice function c_A for any set A, namely if $U \subseteq A$ and $U \neq \emptyset$, choose $c_A(U) = u_0 \in U$ where $F(u_0)$ is the $\in$-least element of $F(U)$.