(1) Working in ZF, show that an infinite well-orderable set is Dedekind infinite. Conclude that an amorphous set is not well-orderable. (Interpret "well-orderable" to mean "equipotent with an ordinal".)

Assume that the bijection $f\colon \alpha \to X$ establishes that X is equipotent with the ordinal α. Since X is infinite, α must be infinite. The ordinals are totally ordered by $\in$, so α must be $\in$-comparable with ω. We do not have $\alpha \in \omega$, since α is infinite, so we must have $\omega = \alpha$ or $\omega \in \alpha$. In either case, $\omega \subseteq \alpha$. The inclusion map $\iota\colon \omega \to \alpha$ composed with f is an injective function $f \circ \iota\colon \omega \to X$, establishing that $|\omega| \leq |X|$, so X is Dedekind infinite.

For the last line of the problem, an amorphous set A is infinite and Dedekind finite, so there can be no bijection $f\colon \alpha \to A$ for any ordinal α.

(2) Do Exercise 6.1.3. (There exist $2^{\aleph_0}$ well-orderings of the set of all natural numbers.)

We have already shown that there exist $2^{\aleph_0}$-many distinct bijections from ω to ω. Each such bijection $\pi\colon \omega \to \omega$ may be used to define a binary relation $<_\pi$ on ω:

$$m <_\pi n \Leftrightarrow \pi^{-1}(m) \in \pi^{-1}(n).$$

With this definition, the bijection $\pi\colon \omega \to \omega$ becomes an order-isomorphism

$$\pi\colon \langle \omega; \in \rangle \to \langle \omega; <_\pi \rangle.$$

Since $\langle \omega; <_\pi \rangle$ is order-isomorphic to $\langle \omega; \in \rangle$, $\langle \omega; <_\pi \rangle$ is a well-ordered set, so $<_\pi$ is a well-ordering of ω.

I claim that distinct bijections π, π' yield distinct order relations $<_\pi, <_{\pi'}$ on ω. Assume instead that $\pi \neq \pi'$, but $<_\pi = <_{\pi'}$. Then π and π' must be distinct isomorphisms from the well-ordered set $\langle \omega; \in \rangle$ to the well-ordered set to $\langle \omega; <_\pi \rangle = \langle \omega; <_{\pi'} \rangle$. This contradicts the fact that isomorphisms between well-ordered sets are unique when they exist.

The set

$$\{<_\pi \in \mathcal{P}(\omega \times \omega) \mid \pi \text{ a permutation of } \omega\}$$

contains $2^{\aleph_0}$-many distinct well-orderings of ω. There cannot be more, since there are only $2^{\aleph_0}$-many binary relations on ω. (The total number of binary relations is $|\mathcal{P}(\omega \times \omega)| = |\mathcal{P}(\omega)| = |2^\omega| = 2^{\aleph_0}$.)

(3) Do Exercise 6.2.8. (If X is a nonempty set of ordinals, then $\bigcap X$ is an ordinal. Moreover, $\bigcap X$ is the least element of X.)

The arguments for this were presented in class duing the October 30 and November 1 lectures.

On October 30, in Lemma 1, Part 4, we showed that the intersection of a class of ordinals is an ordinal. The idea was to show that if each element of X is a transitive set of transitive sets, then $\bigcap X$ is also a transitive set of transitive sets.

On November 1, when showing that ON is well-ordered as a class, we showed that if X is a nonempty subset of ordinals, then the ordinal $\bigcap X$ is the least element X. The idea was this: we know that $\bigcap X \subseteq X_i$ for every $X_i \in X$. Hence either (i) $\bigcap X \subsetneq X_i$ for every $X_i \in X$ or (ii) $\bigcap X = X_i$ for some $X_i \in X$ and $\bigcap X \subsetneq X_j$ for every $X_j \in X, j \neq i$. Using Lemma 2 from October 30, Case (i) leads to $\bigcap X \in X_i$ for every $X_i \in X$, hence to $\bigcap X \in \bigcap X$, which contradicts the Axiom of Foundation. Case (ii) leads to the conclusion that $\bigcap X = X_i$ is the $\in$-least element of X, as desired.