(1) Let A be an amorphous set. Show that $A \times A$ is not amorphous. Conclude that A is an infinite set such that $|A \times A| \neq |A|$.

The fact that A is amorphous means that A is infinite, but it cannot be partitioned into two infinite cells.

Assume that A is amorphous and choose $a \in A$. We intend to argue that (i) $A \times \{a\}$ is an infinite subset of $A \times A$ and (ii) the complementary set $A \times (A - \{a\})$ is also an infinite subset of $A \times A$. This will show that $\Pi = \{A \times \{a\}, A \times (A - \{a\})\}$ is a partition of $A \times A$ into two infinite cells, preventing $A \times A$ from being amorphous.

We establish Claim (i) by observing that $A \times \{a\}$ is equipotent with A, and A is infinite, so $A \times \{a\}$ must be infinite. (One may establish equipotence by observing that the first projection function $\pi_1\colon (A \times \{a\}) \to A\colon (x, a) \mapsto x$ is a bijection.)

We can establish Claim (ii) with a slight modification of the argument in the previous paragraph. The first projection function $\pi_1\colon (A \times (A - \{a\})) \to A\colon (x, a) \mapsto x$ is a surjection from $A \times (A - \{a\})$ onto an infinite set A. By the contrapositive of Theorem 2.5 of HJ[1] this is enough to conclude that $A \times (A - \{a\})$ is infinite.

One may establish Claim (ii) a different way based on Theorem 2.4 of HJ.[2] Argue as follows: A is infinite. If $a \in A$, then $A - \{a\}$ is also infinite according to Question 1 of Quiz 5. Choose $b \in A - \{a\}$. The set $A \times \{b\}$ is infinite by the argument for Claim (i). Since $A \times \{b\}$ is an infinite subset of $A \times (A - \{a\})$, it follows from the contrapositive form of Theorem 2.4 that $A \times (A - \{a\})$ is infinite.

For the final sentence of the problem, we observe that the property of being amorphous is an equipotence invariant: If A is amorphous and $|A| = |B|$, then B is amorphous. To see this, assume instead that $f\colon A \to B$ is a bijection. If $\{B_0, B_1\}$ is a partition of B into two infinite cells, then $\{f^{-1}(B_0), f^{-1}(B_1)\}$ is a partition of A into two infinite cells.

(2) Do Exercise 5.1.7. (Use Cantor's Theorem to show that the "set of all sets" does not exist.)

Assume that the class $\mathcal{S}$ of all sets is a set. Since $\mathcal{P}(\mathcal{S})$ consists of sets, $\mathcal{P}(\mathcal{S}) \subseteq \mathcal{S}$ and therefore the inclusion map witnesses that (i) $|\mathcal{P}(\mathcal{S})| \leq |\mathcal{S}|$ holds. By Cantor's Theorem, (ii) $|\mathcal{S}| < |\mathcal{P}(\mathcal{S})|$ holds. Apply the Canter-Bernstein-Schröder Theorem to (i) and (ii) to obtain $|\mathcal{S}| = |\mathcal{P}(\mathcal{S})|$, which contradicts Cantor's Theorem.

(3) (a) Give an example of a set that is transitive, but not well-ordered by epsilon.

$$T = \mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

T is transitive, since $\emptyset, \{\emptyset\} \in T$. and the only instances of sets R, S, T satisfying $R \in S \in T$ are when $R = \emptyset$ or $R = \{\emptyset\}$.

T is not well-ordered by $\in$, since $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are incomparable under $\in$.

[1] This result states that if the domain a function is finite, then the image is finite. The contrapositive asserts that if the image is infinite, then the domain must be infinite.

[2] This theorem states that a subset of a finite set is finite.

(b) Give an example of a set that is well-ordered by epsilon, but is not transitive.

$$W = \{\{\emptyset\}\}.$$

W is not transitive, since $\emptyset \in \{\emptyset\} \in W$, but $\emptyset \notin W$.

W is well-ordered by $\in$, since W has only one element and any one-element ordered set is well-ordered by $\in$.