(1) Define a binary operation $\circ$ on ω as follows. Given $m, n \in \omega$, choose sets A, B with $|A| = m$, $|B| = n$ and define $m \circ n = |A \times B|$.

(a) Show that $\circ$ is well defined.

(b) Show that $m \circ 0 = 0$ and $m \circ S(n) = (m \circ n) + m$. (That is, $\circ$ satisfies the recursion that defines multiplication.)

(c) Conclude that $m \circ n = mn$. (This shows that $|A \times B| = |A| \cdot |B|$ for finite sets.)

(a) We must show that if $|A| = |A'|$ and $|B| = |B'|$, then $|A \times B| = |A' \times B'|$. For this, assume that $f\colon A \to A'$ is a bijection with inverse f^{-1} and $g\colon B \to B'$ is a bijection with inverse g^{-1}. I claim that

$$f \times g\colon A \times B \to A' \times B'\colon (a, b) \mapsto (f(a), g(b))$$

is a bijection with inverse $f^{-1} \times g^{-1}$. To verify this, observe that

$$\begin{aligned} ((f^{-1} \times g^{-1}) \circ (f \times g))(a, b) &= (f^{-1} \times g^{-1})(f(a), g(b)) \\ &= ((f^{-1} \circ f)(a), (g^{-1} \circ g)(b)) \\ &= (a, b). \end{aligned}$$

A similar calculation shows that $((f \times g) \circ (f^{-1} \times g^{-1}))(a', b') = (a', b')$, so $(f \times g)^{-1} = (f^{-1} \times g^{-1})$. Since $(f \times g)$ is invertible, it is a bijection.

(b) The fact that $m \circ 0 = 0$ follows from the fact t5hat $m \times \emptyset = \emptyset$. To show that $m \circ S(n) = (m \circ n) + m$ holds we should explain why there is a bijection between $m \times S(n) = m \times (n \sqcup \{n\}) = (m \times n) \sqcup (m \times \{n\})$ and $(m \times n) \sqcup m$. The bijection

$$F\colon (m \times n) \sqcup (m \times \{n\}) \to (m \times n) \sqcup m$$

can be taken to be $\mathrm{id}_{m \times n} \cup \pi$ where $\pi\colon (m \times \{n\}) \to m$ is the function defined by $(i, n) \mapsto i$.

(c) Since the circle operation satisfies the recursion that defines multiplication of natural numbers, the circle operation must equal the operation of multiplication of natural numbers according to the uniqueness statement of the Recursion Theorem.

(2) Show that the real line has the same cardinality as the real plane. (Hint: By the CBS Theorem you only need to find 1-1 functions in each direction. For a 1-1 function from the plane to the line, try mapping a point (x, y) in the plane to the real number obtained by interlacing the digits of x and y. Be careful to explain exactly what you mean, noting that some real numbers have more than one decimal representation.).

The function $f\colon \mathbb{R} \to \mathbb{R}^2\colon r \mapsto (r, 0)$ is injective, so $|\mathbb{R}| \leq |\mathbb{R}^2|$.

For the other comparison, following the hint, we will consider the interlacing function $g\colon \mathbb{R}^2 \to \mathbb{R}$. Here I will write a real number as

$$\mathbf{a} = \ldots a_2 a_1 a_0 \bullet a_{-1} a_{-2} \ldots,$$

but this will be done under the assumption that only finitely many of the digits left of the point are nonzero. The map is:

$$(\mathbf{a}, \mathbf{b}) = (\ldots a_1 a_0 \bullet a_{-1} a_{-2} \ldots, \ldots b_1 b_0 \bullet b_{-1} b_{-2} \ldots) \mapsto \ldots a_1 b_1 a_0 b_0 \bullet a_{-1} b_{-1} a_{-2} b_{-2} \ldots = \mathbf{a}\widehat{\ }\mathbf{b}.$$

In order to avoid the type of conflict mentioned in the problem hint, I will assume that the input real numbers $\mathbf{a}$ and $\mathbf{b}$ are written in base 2, so only the digits 0 and 1 occur in their expansions. But I will assume that the output number $\mathbf{a}\widehat{\ }\mathbf{b}$ is considered to be written in base 10. Now, if $(\mathbf{a}, \mathbf{b}) \neq (\mathbf{a}', \mathbf{b}')$, then $\mathbf{a}\widehat{\ }\mathbf{b}$ and $\mathbf{a}'\widehat{\ }\mathbf{b}'$ are different decimal numbers. They cannot represent the same real number, since if they did they would be two different decimal expansions, *neither containing the digit 9*, which represent the same real number. But, when a real number has more than one decimal expansion, at most one of the expansions omits the digit 9.

(3) Show that $|(A^B)^C| = |A^{B\times C}|$.

There is a bijection from $A^{B\times C}$ to $(A^B)^C$ called "currying", which assigns to a function $f\colon B \times C \to A$ the function $g\colon C \to A^B\colon c \mapsto f(x, c)$. The inverse of the function $f \mapsto g$ (which is called "uncurrying") is the function that assigns to any $G\colon C \to A^B$ the function $F\colon B \times C \to A\colon (b, c) \mapsto G(c)(b)$.

One should check that (uncurrying $\circ$ currying) equals the identity and that (currying $\circ$ uncurrying) equals the identity.

Suppose we are given $f\colon B \times C \to A$. Currying f yields $g\colon C \to A^B\colon c \mapsto f(x, c)$. Uncurrying g yields

$$F\colon B \times C \to A\colon (b, c) \mapsto g(c)(b) = f(x, c)(b) = f(b, c),$$

so $F(b, c) = f(b, c)$ for all $(b, c) \in B \times C$. This proves that $F = f$.

Now suppose that we are given $G\colon C \to A^B$ and we uncurry G to $F\colon B\times C \to A\colon (b, c) \mapsto G(c)(b)$. Currying F now yields

$$g\colon C \to A^B\colon c \mapsto F(x, c) = G(c)(x),$$

so $g(c)(x) = G(c)(x)$ for all $x \in B$, so $g(c) = G(c)$ for all $c \in C$. This proves that $g = G$.