In the solutions below I will assume the truth of all the Laws of Addition from the handout arithmetic.pdf. Following a common convention, I may denote the product of $m, n \in \mathbb{N}$ by either mn or by $m \cdot n$. Also, I will use the following abbreviations:

- IC = Initial Condition
- RR = Recurrence Relation
- IH = Inductive Hypothesis

(1) Prove that $(m+n)\cdot k = (m\cdot k) + (n \cdot k)$ for all $m, n, k \in \mathbb{N}$.

We are proving the Right Distributive Law.
This is a proof by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned} (m+n)0 &= 0 && (\text{IC}, \cdot) \\ &= 0 + 0 && (\text{IC}, +) \\ &= m0 + n0 && (\text{IC}, \cdot) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} (m+n)\cdot S(k) &= (m+n)k + (m+n) && (\text{RR}, \cdot) \\ &= (mk + nk) + (m+n) && (\text{IH}) \\ &= mk + (nk + (m+n)) && (\text{Associative Law}, +) \\ &= mk + ((nk+m)+n)) && (\text{Associative Law}, +) \\ &= mk + ((m+nk)+n)) && (\text{Commutative Law}, +) \\ &= mk + (m + (nk+n)) && (\text{Associative Law}, +) \\ &= (mk+m) + (nk+n) && (\text{Associative Law}, +) \\ &= (m \cdot S(k)) + (n \cdot S(k)) && (\text{RR}, \cdot) \end{aligned}$$

(2) Prove that $m \cdot n = n \cdot m$ for all $m, n \in \mathbb{N}$.

Lemma 1. $0k = 0$.

Proof. This is a proof by induction on k.
(Base Case: $k = 0$)

$$00 = 0 \qquad (\text{IC}, \cdot)$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} 0 \cdot S(k) &= 0k + 0 && (\text{RR}, \cdot) \\ &= 0 + 0 && (\text{IH}) \\ &= 0 && (\text{IC}, +) \end{aligned}$$

Lemma 2. $1m = m$.

Proof. This is a proof by induction on m.
(Base Case: $m = 0$)

$$10 = 0 \qquad (\text{IC}, \cdot)$$

(Inductive Step: Assume true for m, prove true for $S(m)$)

$$\begin{aligned} 1 \cdot S(m) &= (1m) + 1 && (\text{RR}, \cdot) \\ &= m + 1 && (\text{IH}) \\ &= S(m) && (\text{Law (a) of Addition}) \end{aligned}$$

Solution to the Problem.
Proof. We prove that $mn = nm$ by induction on n.
(Base Case: $n = 0$)

$$\begin{aligned} m0 &= 0 && (\text{IC}, \cdot) \\ &= 0m && (\text{Lemma 1}) \end{aligned}$$

(Inductive Step: Assume true for n, prove true for $S(n)$)

$$\begin{aligned} m \cdot S(n) &= (mn) + m && (\text{RR}, \cdot) \\ &= (nm) + m && (\text{IH}) \\ &= (nm) + (1m) && (\text{Lemma 2}) \\ &= (n+1) \cdot m && (\text{Exercise 1}) \\ &= S(n) \cdot m && (\text{Law (a) of Addition}) \end{aligned}$$

(3) Prove that $m^{n+k} = m^n \cdot m^k$. (You may need to prove some lemmas first.)

Lemma 1. (Law (d) of Multiplication) $m1 = m$.

Proof.

$$\begin{aligned} m \cdot 1 &= m \cdot S(0) && (\text{Definition of 1}) \\ &= (m \cdot 0) + m && (\text{RR}, \cdot) \\ &= 0 + m && (\text{IC}, \cdot) \\ &= m && (\text{Law (c) of Addition}) \end{aligned}$$

Lemma 2. (Left Distributive Law) $m(n + k) = (mn) + (mk)$.
Proof. We prove this by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned} m(n+0) &= mn && (\text{IC}, +) \\ &= mn + 0 && (\text{IC}, +) \\ &= mn + m0 && (\text{IC}, \cdot) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned}
m \cdot (n + S(k)) &= m \cdot S(n+k) && (\text{RR}, +) \\
&= m(n+k) + m && (\text{RR}, \cdot) \\
&= (mn + mk) + m && (\text{IH}) \\
&= mn + (mk + m) && (\text{Associative Law}, +) \\
&= mn + mS(k) && (\text{RR}, \cdot)
\end{aligned}$$

Lemma 3. (Associative Law of Multiplication) $m(nk) = (mn)k$.

Proof. We prove this by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned}
m \cdot (n \cdot 0) &= m \cdot 0 && (\text{IC}, \cdot) \\
&= 0 && (\text{IC}, \cdot) \\
&= (m \cdot n) \cdot 0 && (\text{IC}, \cdot)
\end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned}
m \cdot (n \cdot S(k)) &= m(nk + n) && (\text{RR}, \cdot) \\
&= m(nk) + mn && (\text{Left Distributive Law, Lemma 2}) \\
&= (mn)k + mn && (\text{IH}) \\
&= (mn)k + (mn) \cdot 1 && (\text{Law (d) of Multiplication, Lemma 1}) \\
&= (mn)(k+1) && (\text{Left Distributive Law, Lemma 2}) \\
&= (mn)S(k) && (\text{Law (a) of Addition})
\end{aligned}$$

Solution to the Problem.
Proof. We prove that $m^{n+k} = m^n \cdot m^k$ by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned}
m^{n+0} &= m^n && (\text{IC}, +) \\
&= m^n \cdot 1 && (\text{Law (d) of Multiplication, Lemma 1}) \\
&= m^n \cdot m^0 && (\text{IC}, \text{Exp})
\end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned}
m^{n+S(k)} &= m^{S(n+k)} && (\text{RR}, +) \\
&= m^{n+k} \cdot m && (\text{RR, Exp}) \\
&= (m^n \cdot m^k) \cdot m && (\text{IH}) \\
&= m^n \cdot (m^k \cdot m) && (\text{Associative Law}, \cdot; \text{Lemma 3}) \\
&= m^n \cdot m^{S(k)} && (\text{RR, Exp})
\end{aligned}$$