(1) Do Exercise 2.3.9 (a). (Show that the set B^A exists.) (If you use the hint, show that it is true. Definition 2.3.13 is relevant to this problem.)

B^A is meant to denote the set of all functions $f\colon A \to B$. Each such function f is a binary relation from A to B, hence is a subset of $A \times B$, hence is an element of $\mathcal{P}(A \times B)$. This shows that $B^A \subseteq \mathcal{P}(A \times B)$.

To show that B^A is a set we explain how to construct it. We could define it using the Axiom of Separation by

$$B^A = \{f \in \mathcal{P}(A \times B) \mid \varphi_{\text{function}}(f, A, B)\}$$

if we had a formula $\varphi_{\text{function}}(x, y, z)$ expressing the property that x is a function from y to z. This formula needs to say that $x \subseteq y \times z$ and that x satisfies the function rule: for every $u \in y$ there is a unique $v \in z$ such that $(u, v) \in x$.

Assuming that we have already constructed formulas for the less-complex concepts

1. $\varphi_{\text{binary relation}}(x, y, z)$: "$x \subseteq y \times z$"
2. $\varphi_{\text{pair}}(p, u, v)$: "$p = (u, v)$"

we could express $\varphi_{\text{function}}(x, y, z)$ as

$$\varphi_{\text{binary relation}}(x, y, z) \wedge (\forall u)((u \in y) \to (\exists! v)((v \in z) \wedge ((\exists p)(\varphi_{\text{pair}}(p, u, v) \wedge (p \in x))))).$$

Here $\exists!$ means "there exists a unique". We write $(\exists! s)Q(s)$ to abbreviate

$$(\exists t)Q(t) \wedge (\forall r)(\forall s)((Q(r) \wedge Q(s)) \to (r = s)).$$

(2) Do Exercise 3.2.6. (Prove that each natural number is the set of all smaller natural numbers, i.e., $n = \{m \in \mathbb{N} \mid m < n\}$.)

Let $\varphi(n)$ be the statement that $n \subseteq \mathbb{N}$.

1. (Basis of Induction) $\varphi(0)$ holds since $0 = \emptyset$ is a subset of every set.
2. (Inductive Step) Assume that $\varphi(k)$ holds for some $k \in \mathbb{N}$, i.e., $k \subseteq \mathbb{N}$. Since $k \in \mathbb{N}$, we have $\{k\} \subseteq \mathbb{N}$. Since both k and $\{k\}$ are subsets of $\mathbb{N}$, so is $k \cup \{k\} = S(k)$. This shows that $k \subseteq \mathbb{N}$ implies that $S(k) \subseteq \mathbb{N}$.

Now we argue that any natural number n is the set of its $<$-predecessors (in symbols, this is the assertion $n = \{m \in \mathbb{N} \mid m < n\}$). We use that we have defined the order on $\mathbb{N}$ so that for $m, n \in \mathbb{N}$ we have $m < n$ iff $m \in n$. (Thus, $<$ on $\mathbb{N}$ equals the restriction to $\mathbb{N}$ of the relation $\in$.)

(a) If $x \in n$, then $x \in \mathbb{N}$ by the inductive argument above, hence $x < n$ by the definition of $<$ on $\mathbb{N}$.

(b) If $x < n$, then according to the definition of $<$ we have $x \in n$.

Items (a) and (b) + the Axiom of Extensionality prove that $n = \{m \in \mathbb{N} \mid m < n\}$.

(3) Do Exercise 3.2.8. (Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) > f(n+1)$.)

One may use the Axiom of Foundation to prove the more general fact that, for any set A, there is no function $f : \mathbb{N} \to A$ such that for all $n \in \mathbb{N}$, $f(n+1) \in f(n)$. That argument goes as follows: if f were such a function, then its image $\text{im}(f)$ is a nonempty set that has no $\in$-minimal element.

But we can prove the statement of Exercise 3.2.8 without using the Axiom of Foundation, since the image of f consists of natural numbers. We just observe that $\text{im}(f)$ is a nonempty subset of $\mathbb{N}$ that has no $<$-least element, contradicting the fact that $\mathbb{N}$ is well-ordered by $<$.