(1) How many equivalence relations on the set $\{0, 1, 2\}$ are there? How many partial orderings on $\{0, 1, 2\}$ are there? (To answer this, just write them down or draw the appropriate picture. You don't have to prove that your lists are complete, but to get full credit your lists must be complete. You may assume anything about counting that you learned in grade school, even if we haven't proved it yet.)

For the first part of the problem (equivalence relations), I will count the equivalence relations according to the number of equivalence classes.

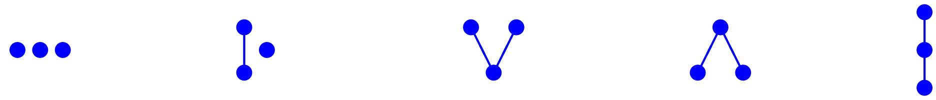
1. (3 equivalence classes.) The associated partition can only be 0/1/2. There is one equivalence relation of this type, namely $\{(0, 0), (1, 1), (2, 2)\}$.
2. (2 equivalence classes) One class must be of size 1 and the other class must be of size 2. The associated partition must be one of 0/12, 1/02, or 2/01. There are three ways to choose the class of size 1, and once you choose it the other class is determined. Therefore there are three equivalence relations of this type: $\{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$, $\{(0, 0), (1, 1), (2, 2), (0, 2), (2, 0)\}$, and $\{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$.
3. (1 equivalence class) The associated partition can only be 012 (all elements in one cell). This equivalence relation is

$$\{0, 1, 2\} \times \{0, 1, 2\} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\},$$

the "full" equivalence relation, or the "universal" equivalence relation.

This yields a total of $1 + 3 + 1 = 5$ equivalence relations on a 3-element set.

For the second part of the problem, the "shape" or "type" of a 3-element ordered set must be one of these five:



If you label the dots with "0", "1" and "2" in essentially different ways, you should find that there is only 1 partial order of the first type, 6 of the second type, 3 of the third type, 3 of the fourth type, and 6 of the fifth type. The final answer is $1 + 6 + 3 + 3 + 6 = 19$.

(2) Recall that a partial ordering of a set is a binary relation. What are the least and largest number of pairs that can occur in a partial ordering of an n-element set? (Problem to think about: is every intermediate value equal to the number of pairs of some partial ordering of an n-element set?)

A partial order on $X = \{x_1, x_2, \ldots, x_n\}$ must contain the diagonal subset

$$D = \{(x_1, x_1), \ldots, (x_n, x_n)\} \subseteq X \times X,$$

so at least n pairs are required. Thus, n is a lower bound for the number of pairs required for a partial order of an n-element set. Since D is already a partial order on X, there do exist partial orders with only n pairs. This shows that the least number of pairs is exactly n.

The number of pairs in the total ordering $x_1 \le x_2 \le \cdots \le x_n$ can be counted as follows: for each x_i, $i = 1, \ldots, n$, count the number of pairs of the form $(\_, x_i)$ that are contained in the relation. The result will be $1 + 2 + \cdots + n = n(n+1)/2$. This is a lower bound for the largest number of pairs that can be realized, since this number of pairs can be realized. On the other hand, it is impossible to find a partial order on X with more pairs, for the following reason. By reflexivity, any partial order must contain the n diagonal pairs (x_i, x_i), $i = 1, \ldots, n$. The relation can contain at most half of the remaining $n^2 - n$ pairs of the form $(u, v), u \neq v$, since (by antisymmetry) the relation cannot contain both $u \le v$ and $v \le v$ when $u \neq v$. Thus, a partial order can contain at most $n + \frac{1}{2}(n^2 - n) = n(n+1)/2$. Since we have an argument that no partial order on X can contain more than $n(n+1)/2$ pairs, and we know that a total order contains this many pairs, the largest number of pairs is exactly $n(n+1)/2$.

Concerning the "Problem to think about", the answer is "Yes". To see this, assume that X has n elements, $R \subseteq X \times X$ is a partial order on X, and R has k pairs for some k satisfying $n \le k < n(n+1)/2$. Let's argue that it is possible to add one additional pair to R to create a set $R' = R \cup \{(a, b)\}$ so that R' is a partial ordering on X and R' has $k+1$ elements.

Since R has $k < n(n+1)/2$ pairs, R is not a total order on X. Choose $a \in X$ that is minimal in the R-order for the property that a is R-incomparable with at least one element of X. Now choose $b \in X$ maximal among the elements that are R-incomparable with a. Check that the relation $R' = R \cup \{(a, b)\}$ is reflexive, antisymmetric, and transitive and has exactly $k+1$ pairs.

Starting with a partial ordering R_n of X that has n pairs, one can use the above construction over and over to create a sequence of partial orderings of X, $R_n, R_{n+1}, \ldots, R_{n(n+1)/2}$, where R_k contains exactly k pairs.

(3) Do Exercise 3.2.1:

Let $n \in \mathbb{N}$. Prove that there is no $k \in \mathbb{N}$ such that $n < k < n+1$.

We will show the more general fact that for any set x, there is no set y such that $x \in y \in S(x)$.

If $y \in S(x)$, then either (i) $y \in x$ or (ii) $y = x$. If we also have $x \in y$, then in Case (i) we have

$$\cdots \in y \in x \in y \in x,$$

while in Case (ii) we have

$$\cdots \in x \in x \in x \in x.$$

In either case, the unordered pair $\{x, y\}$ has no $\in$-minimal element, contradicting the Axiom of Foundation.