1. Using Zorn's Lemma, show that every connected graph has a spanning tree.

Let $G = \langle V; E\rangle$ be a graph with vertex-set V and edge-set E. Let's agree to call a subgraph $H = \langle V_H; E_H\rangle$ with $V_H = V$ and $E_H \subseteq E$ a 'subforest' if it is acyclic, and a 'spanning tree' if it is acyclic and connected. That is, a subforest or spanning tree will have the same vertices as G and will have a subset of the edges of G, both are required to be acyclic, but a spanning tree is also required to be connected.

Define a poset $\langle P; <\rangle$ whose elements are edge-sets E_H where H is a subforest of G, and the set P is ordered by $E_H < E_K$ iff $E_H \subsetneq E_K$ (the proper inclusion order).

Claim. $\langle P; <\rangle$ is inductively ordered.

Proof. We argue that if $C \subseteq P$ is a chain, then $\bigcup C \in P$. Since each element of C is the edge-set of a subforest, $\bigcup C$ is a set of edges. To see that $\bigcup C \in P$, we must argue that $\bigcup C$ is acyclic. If not, then there is a cycle $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ with $\{v_i, v_{i+1}\} \in \bigcup C$ for all i. (I will write undirected edges as doubletons.) For each $i = 1, \ldots, n$, there exists $E_i \in C$ with $(v_i, v_{i+1}) \in E_i$. Since C is a chain and is ordered by inclusion, there is an $i_0 \in \{1, 2, \ldots, n\}$ such that E_{i_0} contains E_i for all $i = 1, \ldots, n$, and that means E_{i_0} contains the cycle $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$. This is impossible, since $E_{i_0} \in C$ and all sets in C are edge-sets of subforests. The conclusion is that $\bigcup C$ is acyclic, hence $\bigcup C \in P$, hence $\bigcup C$ is an upper bound to C in P. □

By the Claim, together with Zorn's Lemma, there must exist a maximal element E_H of P. The subgraph $H = \langle V; E_H\rangle$ is a subforest of G whose edge-set is maximal in the ordering of P. This maximality implies that adding any single new edge to E_H will result in a cycle. We need to argue that $H = \langle V; E_H\rangle$ is a spanning tree for G. For this, we need to show that H is connected.

To complete the proof, assume that H is not connected. This means that there exist $u, w \in V$ which are not connected by any path in H. Since G is connected, the vertices u and w are connected by a path in G, say a path of length m: $u = u_1, \ldots, u_m = w$. Among all such choices of u and w satisfying these conditions, we may assume that our choices have been made to minimize m. Necessarily $m = 1$, as I now explain. If $u = u_1, \ldots, u_m = w$ with $m > 1$ and m chosen minimally, then u_2 is connected to both u and w in G, but cannot be connected to both u and w in H. This means that there are G-paths $u - -u_2$ and $u_2 - -w$ that are both of length $< m$ and they are not both H-paths, so one of these paths contradicts the minimality of m.

So far we have learned that H has vertices u and w that are adjacent (= connected by a path of length $m = 1$) in G but not connected by any path in H. In particular, this means that $\{u, w\} \notin E_H$. Let $E_{H'} = E_H \cup \{\{u, w\}\}$. The maximality of E_H implies that $H' := \langle V; E_{H'}\rangle$ has a cycle. Since $H = \langle V; E_H\rangle$ did not have a cycle, the cycle in H' must contain the new edge $\{u, w\}$. Assume that the cycle is $u, w = w_1, \ldots, w_k = u$, where we may assume that the vertices are distinct except for the first and last. But now $w = w_1, \ldots, w_k = u$ is a path in H that connects w to u, and we chose these two elements so that there was no such path. This contradicts our assumption that H is not connected. The proof is complete.

2. (Exercise 9.1.10.) Prove that $\kappa \cdot \kappa \cdots$ (λ times) $= \kappa^\lambda$.

The lefthand side of

$$\kappa \cdot \kappa \cdots \stackrel{?}{=} \kappa^\lambda$$

is the unique cardinal equipotent with the set $A = \prod_{\alpha<\lambda} \kappa$, while the righthand side is the unique cardinal that is equipotent with the set $B = \{f : \lambda \to \kappa \mid f \text{ is a function}\}$. To prove the cardinals are equal, it suffices to produce a bijection $F \colon B \to A$. One such bijection is

$$B \to A \colon f \mapsto (f(0), f(1), f(2), \ldots).$$

If you prefer to give a bijection in the other direction, i.e. $G \colon A \to B$, then you might choose

$$A \mapsto B \colon t \mapsto \widehat{t}, \quad \text{where} \quad \widehat{t}(\alpha) = t_\alpha.$$

3. Show that if α is any ordinal, then there is an ordinal β of countable cofinality satisfying $\beta > \alpha$.

If α is an ordinal and $\beta := \alpha + \omega$, then $\beta > \alpha$ and β has countable cofinality (since $S = \{\alpha + n \mid n \in \omega\}$ is a countable cofinal subset of β).