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Now our conjecture $1 + 3 + \cdots + (2n + 1) = n^2$ is expressible in a first-order way, $F(n) = n^2$, so the Principle of Induction may be applied to prove the conjecture.

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2. The Principle of Induction allows us to give a finite-length argument to derive conclusions in infinitely many cases.
3. Induction is well-suited to prove facts about recursively-defined objects. (Often it is the only way to prove such facts.)