The Order on $\mathbb{N}$

$m < n \Leftrightarrow m \in n$

$m < n \iff m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

$m < n \iff m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$

$m < n \iff m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$,

$m < n \Leftrightarrow m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

$m < n \Leftrightarrow m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n) \quad$ iff $\quad k < n$ or $k = n \quad$ iff $\quad k \leq n$.
2. For all $n \in \mathbb{N}, \quad 0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n) \quad \text{iff} \quad k < n$ or $k = n \quad \text{iff} \quad k \leq n$.
2. For all $n \in \mathbb{N}, \quad 0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n) \quad \text{iff} \quad k < n$ or $k = n \quad \text{iff} \quad k \leq n$.
2. For all $n \in \mathbb{N}, \quad 0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n) \quad \text{iff} \quad k < n$ or $k = n \quad \text{iff} \quad k \leq n$.
2. For all $n \in \mathbb{N}, \quad 0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n.

$m < n \iff m \in n$

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n) \quad$ iff $\quad k < n$ or $k = n \quad$ iff $\quad k \leq n$.
2. For all $n \in \mathbb{N}, \quad 0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction, so $S(m) \in S(k)$,

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction, so $S(m) \in S(k)$, so $S(m) \leq S(k)$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction, so $S(m) \in S(k)$, so $S(m) \leq S(k)$. If $m = k$, then $S(m) \leq S(k)$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction, so $S(m) \in S(k)$, so $S(m) \leq S(k)$. If $m = k$, then $S(m) \leq S(k)$. In either case, the statement holds for $n = S(k)$.

Let's write $m \leq n$ to mean $m < n$ or $m = n$.

Lemma.

1. For all $k, n \in \mathbb{N}$, $k < S(n)$ iff $k < n$ or $k = n$ iff $k \leq n$.
2. For all $n \in \mathbb{N}$, $0 \leq n$.
3. If $m < n$, then $S(m) \leq n$.

Proof.
The first item can be proved without induction. (Just use $S(n) = n \cup \{n\}$.)

For the second item, prove $\varphi(n) : 0 \leq n$ by induction.

The third item can be proved by induction on n. The Basis of Induction $(n = 0)$ holds vacuously. For the Inductive Step, assume that the statement holds for some $n = k$ and that $m \in S(k) = k \cup \{k\}$. If $m \in k$, then $S(m) \leq k$ by induction, so $S(m) \in S(k)$, so $S(m) \leq S(k)$. If $m = k$, then $S(m) \leq S(k)$. In either case, the statement holds for $n = S(k)$. □

Transitive sets

Transitive sets

Definition.

Transitive sets

Definition. A set T is *transitive* if

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction)

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction)

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k)$$

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

implies $A \in B \in k$

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

implies $A \in B \in k$ (in which case $A \in k$, by IH)

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

implies $A \in B \in k$ (in which case $A \in k$, by IH) or $A \in B = k$.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

   $$A \in B \in S(k) = k \cup \{k\}$$

   implies $A \in B \in k$ (in which case $A \in k$, by IH) or $A \in B = k$. In either case, $A \in k$, so $A \in k \cup \{k\} = S(k)$.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

implies $A \in B \in k$ (in which case $A \in k$, by IH) or $A \in B = k$. In either case, $A \in k$, so $A \in k \cup \{k\} = S(k)$. □

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.
Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

$$A \in B \in S(k) = k \cup \{k\}$$

implies $A \in B \in k$ (in which case $A \in k$, by IH) or $A \in B = k$. In either case, $A \in k$, so $A \in k \cup \{k\} = S(k)$. □

Corollary.

Transitive sets

Definition. A set T is *transitive* if

$$R \in S \in T \Rightarrow R \in T.$$

Theorem. Every natural number is transitive.

Proof. (Induction.)

1. (Basis of Induction) 0 is transitive. (Reason?)
2. (Inductive Step) Assume that k is transitive. Let's check that $S(k)$ is transitive:

   $$A \in B \in S(k) = k \cup \{k\}$$

   implies $A \in B \in k$ (in which case $A \in k$, by IH) or $A \in B = k$. In either case, $A \in k$, so $A \in k \cup \{k\} = S(k)$. □

Corollary. The relation $<$ is transitive on $\mathbb{N}$.

Irreflexivity

Irreflexivity

Theorem.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction)

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction)

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \nless n$ holds for all n by induction.

1. (Basis of Induction) $0 \nless 0$. (Reason?)
2. (Inductive Step) Assume that $k \nless k$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k)$

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$,

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$,

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$. □

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$. $\square$

Corollary.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$. $\square$

Corollary. The relation $<$ is irreflexive and transitive on $\mathbb{N}$, hence is a strict order on $\mathbb{N}$.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$. $\square$

Corollary. The relation $<$ is irreflexive and transitive on $\mathbb{N}$, hence is a strict order on $\mathbb{N}$.

Comment.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \nless n$ holds for all n by induction.

1. (Basis of Induction) $0 \nless 0$. (Reason?)
2. (Inductive Step) Assume that $k \nless k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \nless k$. $\square$

Corollary. The relation $<$ is irreflexive and transitive on $\mathbb{N}$, hence is a strict order on $\mathbb{N}$.

Comment. The Axiom of Foundation provides a simpler proof.

Irreflexivity

Theorem. The relation $<$ is irreflexive on $\mathbb{N}$.

Proof. We prove that $n \not< n$ holds for all n by induction.

1. (Basis of Induction) $0 \not< 0$. (Reason?)
2. (Inductive Step) Assume that $k \not< k$. Assume that $S(k) < S(k) = k \cup \{k\}$. Then $k \in S(k) \in k \cup \{k\}$, so $k \in k$, contrary to $k \not< k$. □

Corollary. The relation $<$ is irreflexive and transitive on $\mathbb{N}$, hence is a strict order on $\mathbb{N}$.

Comment. The Axiom of Foundation provides a simpler proof. If $n < n$, then $\{n\}$ has no $\in$-minimal element.

Trichotomy

Trichotomy

Theorem.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$,

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction)

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction)

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma:

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$. If $m \leq k$, then $m < k \cup \{k\} = S(k)$.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$. If $m \leq k$, then $m < k \cup \{k\} = S(k)$. If $k < m$, then $S(k) \leq m$ by the earlier lemma.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$. If $m \leq k$, then $m < k \cup \{k\} = S(k)$. If $k < m$, then $S(k) \leq m$ by the earlier lemma. □

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$. If $m \leq k$, then $m < k \cup \{k\} = S(k)$. If $k < m$, then $S(k) \leq m$ by the earlier lemma. □

Corollary.

Trichotomy

Theorem. For any $m, n \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

Proof. We prove by induction on n the statement that, for all $m \in \mathbb{N}$, $m < n$ or $m = n$ or $n < m$.

1. (Basis of Induction) The statement is true for $n = 0$ by the earlier lemma: $(0 \leq n)$.
2. (Inductive Step) Assume that k is comparable with all natural numbers. We must show that $S(k)$ is comparable with any $m \in \mathbb{N}$. If $m \leq k$, then $m < k \cup \{k\} = S(k)$. If $k < m$, then $S(k) \leq m$ by the earlier lemma. □

Corollary. The relation $<$ is a strict total order on $\mathbb{N}$.

Well-order

Definition.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not.

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid ; (\forall q \in Q)(b < q)\}.)$

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid ; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid ; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid ; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts:

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$,

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$. $\square$

Well-order

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$. □

Comment.

Well-order

Definition. A partially ordered set $\langle P; < \rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; < \rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$. □

Comment. The Axiom of Foundation provides a simpler proof.

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$. $\square$

Comment. The Axiom of Foundation provides a simpler proof. If
$\emptyset \neq Q \subseteq \mathbb{N}$,

Definition. A partially ordered set $\langle P; <\rangle$ is a *well-order* if every nonempty subset of P has a $<$-least element.

Theorem. $\langle \mathbb{N}; <\rangle$ is a well-order.

Proof. Assume not. Let $Q \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has no $<$-least element. Let $B \subseteq \mathbb{N}$ be the set of all strict lower bounds of Q.
$(B = \{b \in \mathbb{N} \mid; (\forall q \in Q)(b < q)\}.)$

1. $B \subseteq \mathbb{N}$.
2. $0 \in B$.
3. B is closed under successor.

Hence $B = \mathbb{N}$. But this contradicts: $B, Q \subseteq \mathbb{N}$, $B \cap Q = \emptyset$ and $Q \neq \emptyset$. $\square$

Comment. The Axiom of Foundation provides a simpler proof. If $\emptyset \neq Q \subseteq \mathbb{N}$, then any $\in$-minimal element of Q is a $<$-least element.