Infinite versus Dedekind Infinite

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We will prove that "infinite = Dedekind infinite" in ZFC, but the example of amorphous sets shows that the Axiom of Choice cannot be avoided.

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