Counting Problems

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In general, the Sum Rule is suggested when (exclusive) "OR" is being counted, while the Product Rule is suggested when (independent) "AND" is being counted.

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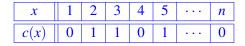
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x	1	2	3	4	5	•••	n
c(x)	0	1	1	0	1	•••	0

There are 2 choices for c(1), 2 (independent) choices for c(2), ..., 2 choices for c(n), so $|X| = 2 \cdot 2 \cdot \cdot \cdot 2 = 2^n$.

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Here we used the fact that a subset $S \subseteq X$ can be described by specifying whether $1 \in S$ AND specifying whether $2 \in X$, etc.

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