Semilattice Modes I: the associated semiring *

Keith A. Kearnes †

Abstract

We examine idempotent, entropic algebras (modes) which have a semilattice term. We are able to show that any variety of semilattice modes has the congruence extension property and is residually small. We refine the proof of residual smallness by showing that any variety of semilattice modes of finite type is residually countable. To each variety of semilattice modes we associate a commutative semiring satisfying $1 + r = 1$ whose structure determines many of the properties of the variety. This semiring is used to describe subdirectly irreducible members, clones, subvariety lattices, and free spectra of varieties of semilattice modes.

1 Introduction

This paper is a companion to [6] which analyzes the structure of finite modes and locally finite varieties of modes. A **mode** is an **idempotent**, **entropic** algebra. That is to say that it is an algebra $\mathbf{A}$ whose basic operations satisfy the idempotent law: $f(x, x, \ldots, x) = x$ and the entropic law: for $f \in \mathrm{Clo}_m\mathbf{A}$ and $g \in \mathrm{Clo}_n\mathbf{A}$ and an $m \times n$ array of elements of A,

$$\begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^n \\ u_2^1 & u_2^2 & & \\ \vdots & & \ddots & \\ u_m^1 & u_m^2 & & u_m^n \end{bmatrix},$$

we have $f(\overline{g(\bar{u}_i)}) = g(\overline{f(\bar{u}^j)})$. This says that f and g **commute** on $[u_i^j]$. The idempotent and entropic laws are equational laws, so any mode generates a variety of modes. A mode is a **semilattice mode** if some binary term interprets as a semilattice operation. A mode is a **Mal'cev mode** if some ternary term interprets as a Mal'cev operation:

$$p(x, y, y) = x = p(y, y, x).$$

Mal'cev modes are also called **affine modes** since every affine algebra has a Mal'cev operation and every Mal'cev mode is affine. We use the names "affine mode" and "Mal'cev mode" interchangeably. The book [10] is a general reference for modes.

The principal result of [6] is the proof that if $\mathbf{A}$ is a finite mode and ρ is the solvable radical of $\mathbf{A}$, then $\mathbf{A}/\rho$ is a semilattice mode and each ρ-class is the universe

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of a solvable submode. It is further shown that a finite solvable mode is the direct product of an affine mode and a strongly solvable mode. The definition of "strongly solvable" can be found in [2]; however, for finite modes we can explain strongly solvable inductively as follows. If $\mathbf{A}$ is a finite mode and α is a minimal congruence on $\mathbf{A}$, then $\mathbf{A}$ is strongly solvable if and only if $\mathbf{A}/\alpha$ is strongly solvable and each α-class is the universe of a subalgebra of $\mathbf{A}$ which is term equivalent to a set with no operations. Thus, a finite simple mode is strongly solvable if it is term equivalent to a set. A mode whose congruence lattice has height two is strongly solvable if it is set-by-set: i.e., if it has a congruence where the factor algebra is a set and each congruence block is a set. We know very little about strongly solvable modes. On the other hand, we know a great deal about affine modes: they are precisely the idempotent reducts of modules over commutative rings. In this paper we explore some properties of semilattice modes. Although our interest in semilattice modes is stimulated by the results of [6], we shall have no need to assume local finiteness in this paper.

There is an unexpected similarity between varieties of semilattice modes and varieties of affine modes. The strongest common link between varieties of semilattice modes and varieties of affine modes is that for either type of variety there is a naturally definable commutative semiring which determines most of the properties of the variety. We use this semiring in exploring the categorical and algebraic properties of varieties of semilattice modes. In this paper we investigate mainly residual smallness and congruence extension in varieties of semilattice modes. We also look at clones, subvariety lattices, and free spectra of varieties of semilattice modes. We find results that are nearly identical to the corresponding results for modules (hence for affine modes). In the follow-up paper, *Semilattice Modes II: the amalgamation property*, we find the surprising result that, unlike varieties of modules (or affine modes) it is *not* true that every variety of semilattice modes has the amalgamation property. In *Semilattice Modes II* we give a characterization of the locally finite varieties of modes which have the amalgamation property.

One can extend our results for semilattice modes and affine modes so that they apply to varieties of locally strongly solvable modes which are abelian. But varieties of locally strongly solvable modes which are not abelian are not known to share any of the properties mentioned above. It is known that locally strongly solvable modes need not have the congruence extension property, but beyond this the structure of locally strongly solvable modes remains obscure.

2 Examples

Of course, the best example of a variety of semilattice modes is the variety of semilattices. Other examples of modes appear in [10]. In this section we describe two examples which particularly illuminate the structure of subdirectly irreducible semilattice modes.

Example 1. Let $\mathbf{S} = \langle S; + \rangle$ be an algebra satisfying:

(a) $\mathbf{S}$ is a join-semilattice.

(b) $\mathbf{S}$ has a least element 0.

Let U be a set of endomorphisms of $\mathbf{S}$ satisfying the following conditions:

(i) U is closed under composition and contains the identity endomorphism.

(ii) Members of U commute.

(iii) All members of U are decreasing in the sense that if $f \in U$, then $f(x) \leq x$ (meaning $f(x) + x = x$).

Now define $\mathbf{S}(U)$ to be the algebra $\langle S; b_f(f \in U)\rangle$ where $b_f(x, y) = f(x) + y$. Properties $(a) - (b)$ of $\mathbf{S}$ and $(i) - (iii)$ of U ensure that $\mathbf{S}(U)$ is a mode.

Now assume that $\mathbf{S}$ satisfies in addition

(c) $\mathbf{S}$ has an element u which is the least element in $S - \{0\}$.

Furthermore, suppose that U satisfies

(iv) If $x < y$ in $\mathbf{S}$, then there is an $f \in U$ such that $f(x) = 0 < f(y)$.

We now show that (c) and (iv) ensure that $\mathbf{S}(U)$ is subdirectly irreducible. Assume that $r, s \in S$ and $s \not\leq r$. Then $r < r + s$, so by property (iv) we can find $f \in U$ such that

$$f(r) = 0 < f(r + s) = f(r) + f(s) = f(s).$$

Hence

$$0 = b_f(r, 0) \ \mathrm{Cg}(r, s) \ b_f(s, 0) = f(s) = b_f(s, u) \ \mathrm{Cg}(r, s) \ b_f(r, u) = u.$$

Hence $(0, u) \in \mathrm{Cg}(r, s)$ whenever $r \neq s$ in $\mathbf{S}(U)$. This proves that $\mathrm{Cg}(0, u)$ is the least nonzero congruence of $\mathbf{S}(U)$ and so $\mathbf{S}(U)$ is subdirectly irreducible. What is surprising is that every subdirectly irreducible semilattice mode is term equivalent to $\mathbf{S}(U)$ for some $\mathbf{S}$ satisfying $(a) - (c)$ and some U satisfying $(i) - (iv)$. This we prove in the next section.

Example 2. Let $\mathbf{R}$ be a commutative ring with unit and let S be the set of ideals of $\mathbf{R}$. For $a, b \in S$, define $a \oplus b = a \cap b$. This makes $\mathbf{S} = \langle S; \oplus\rangle$ a join-semilattice with least element R. For each $r \in R$ define a function $f_r : S \to S$ by

$$f_r(a) = \{s \in R \mid rs \in a\}.$$

Let $U = \{f_r \mid r \in R\}$. Then $\mathbf{S}$ satisfies $(a) - (b)$ and U satisfies $(i) - (iii)$ of Example 1. Hence $\mathbf{S}(U)$ is a semilattice mode.

We shall find later in this paper that if $\mathcal{V}$ is a variety of semilattice modes, then there is a *semiring* $\mathbf{R}$ associated with $\mathcal{V}$ such that every subdirectly irreducible member of $\mathcal{V}$ is equivalent to a subalgebra of $\mathbf{S}(U)$ where $\mathbf{S}$ is the semilattice of annihilator ideals of $\mathbf{R}$ under intersection and $U = R$ (just as we described in the previous paragraph using a ring instead of a semiring). Besides giving us a good description of the subdirectly irreducible members of $\mathcal{V}$, this has two important consequences. First, it shows that varieties of semilattice modes are residually small. Second, it implies severe restrictions on the semilattice order of a subdirectly irreducible semilattice mode of finite type.

3 Residual Smallness

In this section we describe subdirectly irreducible semilattice modes up to term equivalence. From this description we are easily able to deduce that varieties of semilattice modes are residually small.

The statement that an algebra satisfies the idempotent and entropic laws is equivalent to the statement that every term operation commutes with every term and every constant operation. In other words, it means that every term operation commutes with every polynomial operation. We will use without comment the fact that if $\mathbf{A}$ is a mode and $p \in \mathrm{Pol_n}\mathbf{A}$, then $p : \mathbf{A}^n \to \mathbf{A}$ is a homomorphism.

THEOREM 3.1 *If $\mathbf{A}$ is a subdirectly irreducible semilattice mode, then the following are true.*

(i) *$\mathbf{A}$ has an element 0 such that $0 + x = x + 0 = x$ for all $x \in A$. (0 is the least element in the semilattice ordering of A.)*

(ii) *$\mathbf{A}$ has an element u such that $u + x = x + u = x$ for all $x \in A - \{0\}$. (u is the least element in $A - \{0\}$.)*

(iii) *The monolith of $\mathbf{A}$ is the equivalence relation on A having $\{0, u\}$ as the only nontrivial block.*

(iv) *The set $U =$*

$$\begin{aligned}\{s^{\mathbf{A}}(x,0) | s \text{ is a binary term}\} &= \{p(x) \in \mathrm{Pol}_1\mathbf{A} | p(0) = 0\} \\ &= \{q(x) \in \mathrm{Pol}_1\mathbf{A} | 0 \in q(A)\}\end{aligned}$$

is a set of commuting, decreasing endomorphisms of $\langle A; +\rangle$.

(v) *$\mathbf{A}$ is polynomially equivalent to $\langle A; +, U\rangle$*

Proof: Let μ denote the monolith of $\mathbf{A}$ and choose $(a, b) \in \mu - 0_{\mathbf{A}}$. We may assume that $a + b \neq a$. Now consider $e(x) = a + x$. This polynomial is an idempotent endomorphism of $\mathbf{A}$. But $e(a) = a \neq a + b = e(b)$. Hence $\mu = \mathrm{Cg}(a, b) \nsubseteq \ker e$ and so e is 1-1. But the only 1-1, idempotent function on a set is the identity function. This gives us that $a + x = e(x) = x$ ($= x + a$ by the symmetry of $+$) for all $x \in A$. In other words, a is a neutral element for $+$. If $c \in A$ is any element satisfying $c + b \neq c$, then set $f(x) = c + x$. Again we have $f(a) = c \neq c + b = f(b)$, so $f(x)$ is 1-1. This forces $a = f(a) = c$. We conclude that if $c \in A - \{a\}$, then $c + b = c = b + c$. We redenote a and b by 0 and u respectively. The uniqueness of 0 and u (0 is the least element in the semilattice ordering and u is the least element different from 0) implies that μ is the equivalence relation on A generated by $(0, u)$. This establishes (i), (ii) and (iii).

To prove (iv), define $U = \{s^{\mathbf{A}}(x, 0) \in \mathrm{Pol}_1\mathbf{A} \mid s \text{ is a binary term}\}$. If $p(x) = s^{\mathbf{A}}(x, 0) \in U$, then $p(0) = s^{\mathbf{A}}(0, 0) = 0$. Hence $U \subseteq \{p(x) \in \mathrm{Pol}_1\mathbf{A} | p(0) = 0\}$. Now choose any unary polynomial $p(x) = t^{\mathbf{A}}(x, \bar{a})$ such that $p(0) = 0$. Then

$$\begin{aligned}p(x) &= t^{\mathbf{A}}(x, \bar{a}) \\ &= t^{\mathbf{A}}(x + 0, 0 + a_0, \ldots, 0 + a_{n-1}) \\ &= t^{\mathbf{A}}(x, 0, \ldots, 0) + t^{\mathbf{A}}(0, a_0, \ldots, a_{n-1}) \\ &= t_0^{\mathbf{A}}(x, 0) + 0 = t_0^{\mathbf{A}}(x, 0) \in U\end{aligned}$$

where $t_0(x,y)$ is defined to be $t(x,y,\dots,y)$. Hence $\{p(x) \in \mathrm{Pol}_1\mathbf{A} | p(0) = 0\} \subseteq U$ as well. To see that $U = \{q(x) \in \mathrm{Pol}_1\mathbf{A} |\ 0 \in q(A)\}$ observe that if $q \in U$, then $q(0) = 0 \to 0 \in q(A)$. Conversely, if $0 \in q(A)$, then there exists $a \in A$ such that $q(a) = 0$. Since q is order-preserving, $0 \leq q(0) \leq q(a) = 0$ and so $q(0) = 0$. (The reason that q is order-preserving is that each unary polynomial is an endomorphism of $\mathbf{A}$. It follows that each unary polynomial is an endomorphism of the reduct $\langle A; +\rangle$ and therefore is order-preserving.) We now claim that the endomorphisms in U commute with each other and are decreasing with respect to the semilattice order on A. To see that they commute with each other, choose $s^{\mathbf{A}}(x,0), t^{\mathbf{A}}(x,0) \in U$. Then

$$
\begin{aligned}
s^{\mathbf{A}} \circ t^{\mathbf{A}}(x) &= s^{\mathbf{A}}(t^{\mathbf{A}}(x,0),0) \\
&= s^{\mathbf{A}}(t^{\mathbf{A}}(x,0), t^{\mathbf{A}}(0,0)) \\
&= t^{\mathbf{A}}(s^{\mathbf{A}}(x,0), s^{\mathbf{A}}(0,0)) \\
&= t^{\mathbf{A}}(s^{\mathbf{A}}(x,0),0) \\
&= t^{\mathbf{A}} \circ s^{\mathbf{A}}(x).
\end{aligned}
$$

To see that each member of U is decreasing, choose $s^{\mathbf{A}}(x,0) \in U$. Then we have

$$
\begin{aligned}
x + s^{\mathbf{A}}(x,0) &= s^{\mathbf{A}}(x,x) + s^{\mathbf{A}}(x,0) \\
&= s^{\mathbf{A}}(x+x, x+0) \\
&= s^{\mathbf{A}}(x,x) \\
&= x.
\end{aligned}
$$

Hence $s^{\mathbf{A}}(x,0)$ is decreasing. This establishes (iv). We now argue that $\mathbf{A}$ is polynomially equivalent to $\langle A; +, U\rangle$.

Choose an n-ary term t. For each $i < n$ let $f_i(x_i) = t^{\mathbf{A}}(0,0,\dots,x_i,\dots,0)$. Clearly, $f_i(x) \in U$ for each i. The fact that t commutes with $x_0 + x_1 + \cdots + x_{n-1}$ on the array

$$
\begin{bmatrix}
x_0 & 0 & \cdots & 0 \\
0 & x_1 & & 0 \\
\vdots & & \ddots & \\
0 & 0 & & x_{n-1}
\end{bmatrix}
$$

is exactly the statement that

$$t^{\mathbf{A}}(x_0,\dots,x_{n-1}) = \Sigma_{i<n} f_i(x_i).$$

This shows that every term of $\mathbf{A}$ is equal to a polynomial of $\langle A; +, U\rangle$. Consequently, every polynomial of $\mathbf{A}$ is equal to a polynomial of $\langle A; +, U\rangle$. Conversely, each polynomial of $\langle A; +, U\rangle$ is a polynomial of $\mathbf{A}$, since the basic operations of $\langle A; +, U\rangle$ were taken to be polynomials of $\mathbf{A}$. This shows that $\mathbf{A}$ and $\langle A; +, U\rangle$ are polynomially equivalent. $\square$

THEOREM 3.2 *If $\mathbf{A}$ is a subdirectly irreducible semilattice mode with monolith μ and $p \in \mathrm{Pol}_1\mathbf{A}$, then either $p(\mu) \subseteq 0_{\mathbf{A}}$ or else $p(x) = x$ for all $x \in A$*

Proof: As we have seen, $\mathbf{A}$ has elements 0 and u where 0 is the least element in the semilattice ordering of A and u is the least element other than 0. Further, μ is the equivalence relation on A generated by $(0,u)$. If $p(\mu) \not\subseteq 0_{\mathbf{A}}$, then $(p(0), p(u)) \in \mu - 0_{\mathbf{A}}$.

In other words, $\{p(0), p(u)\} = \{0, u\}$. The polynomial p is an endomorphism of $\mathbf{A}$, so it preserves the semilattice order. It follows that $p(0) = 0$, $p(u) = u$. From this and the implication $u \leq x \to u = p(u) \leq p(x)$, we get that $0 < x \to 0 < p(x)$.

We now proceed to show that $p(x) = x$ for all $x \in A$. Assume instead that $w \neq p(w)$ for some $w \in A$. Since $p(0) = 0$, we have from Theorem 3.1 (iv) that $p(x) \in U$. Thus, p is a decreasing join-endomorphism of $\mathbf{A}$. Since $w \neq p(w)$, we get that $p(w) < w$. $\mathbf{A}$ is subdirectly irreducible, so $(0, u) \in \mathrm{Cg}(w, p(w))$. There must be a polynomial $q \in \mathrm{Pol}_1\mathbf{A}$ such that $q(w) \neq q(p(w))$ and one of these elements equals 0. As q is an endomorphism, it preserves the semilattice order and so we must have $0 = q(p(w)) < q(w)$. But $q(p(w)) = p(q(w))$, since both p and q belong to U (both have 0 in their range). If we set $x = q(w)$, we find that $0 < x$ and $0 = p(x)$. This contradicts the final line of the last paragraph. Our assumption that $w \neq p(w)$ for some $w \in A$ is false, so the theorem is proved. □

Theorem 3.2 allows us to refine the characterization of subdirectly irreducible semilattice modes in Theorem 3.1 to a characterization of these modes to within term equivalence.

THEOREM 3.3 $\mathbf{A}$ *is a subdirectly irreducible semilattice mode iff* $\mathbf{A}$ *is term equivalent to an algebra of the form* $\mathbf{S}(U)$ *where* $\mathbf{S}$ *satisfies* $(a)-(c)$ *and* U *satisfies* $(i)-(iv)$ *of Example 1.*

Proof: We prove only the forward implication, since the essential details of the reverse implication can be found in Example 1.

We must construct $\mathbf{S}(U)$ so that it has the same universe and the same terms as $\mathbf{A}$. Begin by choosing S equal to the universe of $\mathbf{A}$. $\mathbf{A}$ is a semilattice mode, so let $x + y$ denote the semilattice operation of $\mathbf{A}$. We have previously proven that $\langle S; +\rangle$ is a join-semilattice which has a least element 0 and an element u which is the least element different from 0. Hence $\mathbf{S}$ satisfies conditions $(a) - (c)$ of Example 1. Let $U = \{s^{\mathbf{A}}(x, 0) |\, s \text{ is a binary term}\}$. The conditions $(i) - (iii)$ from Example 1 follow from Theorem 3.1. Condition (iv) is verified by the same arguments used in the second paragraph of Theorem 3.2 where we showed that if $x = p(w) < w = y$, then there is a $q \in U$ such that $0 = q(x) < q(y)$. Hence this choice of $\mathbf{S}(U)$ yields a subdirectly irreducible mode with the same underlying semilattice as $\mathbf{A}$.

Next, let's show that each term of $\mathbf{S}(U)$ is a term of $\mathbf{A}$ and conversely. For the forward direction, it suffices to show that each $b_f(x, y)$ is the interpretation of a term of $\mathbf{A}$. If $f(x) = s^{\mathbf{A}}(x, 0)$, then

$$b_f(x, y) = s^{\mathbf{A}}(x, 0) + y = s^{\mathbf{A}}(x, 0) + s^{\mathbf{A}}(y, y) = s^{\mathbf{A}}(x + y, 0 + y) = s^{\mathbf{A}}(x + y, y)$$

which is the interpretation of a term of $\mathbf{A}$. For the converse, choose a term $t(x_0, \ldots, x_{n-1})$ of $\mathbf{A}$. We proved (in the last paragraph of Theorem 3.1) that

$$t^{\mathbf{A}}(x_0, \ldots, x_{n-1}) = \Sigma_{i<n} f_i(x_i)$$

for appropriate $f_i \in U$. (We have $f_i \in U$ since $f_i(x_i) = t_i^{\mathbf{A}}(x_i, 0)$ where t_i is defined by $t_i(x_i, y) = t(y, y, \ldots, x_i, \ldots, y)$; x_i is in the i^{th} position.) Since each $f_i \in U$, we have that $f_i(0) = 0$. By Theorem 3.2, we conclude for each $i < n$ that we have (a) $f_i(u) = 0$

or else (b) $f_i(x) = x$ for all $x \in S$. We cannot be in case (a) for all $i < n$ or else we would have

$$u = t^{\mathbf{A}}(u, u, \ldots, u) = \Sigma_{i<n} f_i(u) = \Sigma_{i<n} 0 = 0$$

which is not true. Hence there is an index $l < n$ such that $f_l(x) = x$ for all $x \in S$. Now we have

$$t^{\mathbf{A}}(x_0, \ldots, x_{n-1}) = \Sigma_{i<n} f_i(x_i) = \Sigma_{i \neq l}(f_i(x_i) + x_l) = \Sigma_{i \neq l} b_{f_i}(x_i, x_l).$$

We deduce that $t^{\mathbf{A}}$ is equal to a term of $\mathbf{S}(U)$. This concludes the proof. □

This proof shows that if $\mathbf{A}$ is a subdirectly irreducible semilattice mode and t is a term, then there exists an l for which $\mathbf{A}$ satisfies the equations

$$\begin{aligned} t(x_0, \ldots, x_{n-1}) &= \Sigma_{i<n} f_i(x_i) \\ &= \Sigma_{i \neq l}(f_i(x_i) + x_l) \\ &= \Sigma_{i \neq l} b_{f_i}(x_i, x_l) \\ &= \Sigma_{i \neq l} t_i(x_i, x_l) \end{aligned}$$

where $\mathbf{A} \models b_{f_i}(x_i, x_l) = t_i(x_i, x_l) \stackrel{\text{def}}{=} t(x_l, x_l, \ldots, x_i, \ldots, x_l)$. ($x_i$ in the i^{th} position.) If $\mathbf{A}$ satisfies an equation $t(\bar{x}) = s(\bar{x})$, then by specialization $\mathbf{A} \models t_i(x, y) = s_i(x, y)$ for all i. Thus if l is a value for which $t_l^{\mathbf{A}}(x, 0) = x$, then we also have $s_l^{\mathbf{A}}(x, 0) = x$. It follows that if $\mathbf{A} \models t_i(x, y) = s_i(x, y)$ for all i and l is a value for which $t_l^{\mathbf{A}}(x, 0) = x$, then

$$\begin{aligned} t(\bar{x}) &= \Sigma_{i \neq l} t_i(x_i, x_l) \\ &= \Sigma_{i \neq l} s_i(x_i, x_l) \\ &= s(\bar{x}). \end{aligned}$$

Hence $\mathbf{A} \models t(\bar{x}) = s(\bar{x})$ iff $\mathbf{A} \models t_i(x, y) = s_i(x, y)$ for all i. It follows that the equational theory of a subdirectly irreducible semilattice mode has a basis of equations including the idempotent and entropic laws for all basic operations, equations saying that $+$ is a semilattice operation, and equations involving only binary terms. Let us analyze which binary equations hold in a subdirectly irreducible semilattice mode.

THEOREM 3.4 *Let $\mathbf{A}$ be a subdirectly irreducible semilattice mode. The following are true.*

- (i) $\mathbf{A} \models t(\bar{x}) = s(\bar{x})$ *iff* $\mathbf{A} \models t_i(x, y) = s_i(x, y)$ *for all* i.
- (ii) *For every binary term* $t(x, y)$ *there is an* $f \in U$ *such that* $\mathbf{A} \models t(x, y) = b_f(x, y)$ *or* $\mathbf{A} \models t(x, y) = b_f(y, x)$.
- (iii) $\mathbf{A} \models b_f(x, y) = b_g(x, y)$ *iff* $f = g$. $\mathbf{A} \models b_f(x, y) = b_g(y, x)$ *iff* $f = g = id_A$.

Proof: The proof of (i) precedes the statement of the theorem. The proof of (ii) is contained in the last paragraph of Theorem 3.3. There we show that if t is a term, then $t(x_0, \ldots, x_{n-1}) = \Sigma_{i \neq l} b_{f_i}(x_i, x_l)$. Applied to a binary term, this gives $t(x, y) = b_{f_0}(x, y)$ or $t(x, y) = b_{f_1}(y, x)$. Depending on which case we are in, we let $f = f_0$ or f_1 to get statement (ii). As for statement (iii), assume that $\mathbf{A} \models b_f(x, y) = b_g(x, y)$. Then

$$f(x) = b_f(x, 0) = b_g(x, 0) = g(x).$$

Conversely, if $f = g$, then clearly $b_f(x,y) = b_g(x,y)$. Now assume that $\mathbf{A} \models b_f(x,y) = b_g(y,x)$. Then

$$f(x) = b_f(x,0) = b_g(0,x) = g(0) + x = x = id_A(x).$$

Symmetrically, $g(x) = x = id_A(x)$. Conversely, if $f = g = id_A$, then $b_f(x,y) = x + y = b_g(y,x)$. This finishes the proof. □

In particular, it follows from this theorem that, for modes of the form $\mathbf{S}(U)$, the only binary terms up to equivalence are the obvious ones: the basic operations (of the form $b_f(x,y)$, $f \in U$) and those obtained from them by permuting the two variables. Furthermore, there is no non-obvious equality of binary terms. We will need Theorem 3.4 in the proof of Theorem 3.8 which is one of the main results of this section. Although it is something of a digression at this point, let us probe further into the significance of binary equations for semilattice mode varieties before proceeding. It is not true that all mode varieties are axiomatizable by binary equations. To see this, it suffices to observe that the variety of semilattices is not axiomatizable by binary equations. (Indeed, the variety of semilattices is locally finite; but if $\mathcal{U}$ is the variety of groupoids axiomatized by the binary equations satisfied by the variety of semilattices, then $\mathbf{F}_{\mathcal{U}}(3)$ is infinite.) Nevertheless, it is true that every variety of semilattice modes is axiomatized by the entropic laws together with some binary equations. We prove this in Theorem 3.6 and at the same time we refine Theorem 3.4. First we need to introduce a notation for terms.

If $t(x_0,\ldots,x_{n-1})$ is an n-ary term, then we define $\hat{t}_i(x,y)$ to be $t(y,y,\ldots,x,\ldots,y)+y$ where x is in the i^{th} position only. We will call $\hat{t}_i$ the **$\mathbf{i}^{\text{th}}$ coefficient of** t or **the coefficient of** x_i **in** t and write

$$\hat{t}_0 \bullet x_0 + \cdots + \hat{t}_{n-1} \bullet x_{n-1}$$

as an alternate expression for $t(x_0,\ldots,x_{n-1})$. We call the displayed expression **the coefficient representation for** t. If $\hat{t}_i(x,y) = y$ is an equation of $\mathcal{V}$ for some i, then we will say that the i^{th} coefficient of t **equals 0**. If $\hat{t}_i(x,y) = x + y$ is an equation of $\mathcal{V}$ for some i, then we will say that the i^{th} coefficient of t **equals 1**. We are not claiming that the displayed expression has any meaning other than as an alternate way to refer to $t(x_0,\ldots,x_{n-1})$. We prove a simple lemma about allowable manipulations of coefficient representations. (Beware: There is some abuse of notation in this lemma and throughout this paper concerning coefficient representations. When working within a fixed variety $\mathcal{V}$ of semilattice modes we often say that a binary term $r(x,y)$ "<u>is</u> the i^{th} coefficient" of some term $t(\bar{x})$ when in fact $r(x,y)$ is only $\mathcal{V}$-equivalent to the i^{th} coefficient of $t(\bar{x})$. For all of our purposes it will suffice to know a coefficient only up to $\mathcal{V}$-equivalence.)

LEMMA 3.5 *Let $\mathcal{V}$ be a variety of semilattice modes and let $s(x_0,\ldots,x_{n-1})$ and $t(x_0,\ldots,x_{n-1})$ be n-ary terms. Then the following are true.*

(*i*) *The coefficient representation for $t(x_1,x_0,x_2,\ldots,x_{n-1})$ is*

$$\hat{t}_1 \bullet x_0 + \hat{t}_0 \bullet x_1 + \cdots + \hat{t}_{n-1} \bullet x_{n-1}.$$

(*ii*) *The i^{th} coefficient of $s(\bar{x}) + t(\bar{x})$ is $\hat{s}_i + \hat{t}_i$.*

(*iii*) *The coefficient of* x_0 *in* $s(t(x_0,\ldots,x_{n-1}),y_2\ldots,y_{n-1})$ *is* $\hat{s}_0(\hat{t}_0(x,y),y)$. *(We will write this composition as simply* $\hat{s}_0\hat{t}_0$.*)*

(*iv*) *The coefficient of* x *in* $s(x,x,x_2,\ldots,x_{n-1})$ *is* $\hat{s}_0+\hat{s}_1$.

(*v*) *If* $\mathbf{A}\in\mathcal{V}$ *has a least element 0, then*

$$s^{\mathbf{A}}(x_0,\ldots,x_{n-1})=\hat{s}_0^{\mathbf{A}}(x_0,0)+\cdots+\hat{s}_{n-1}^{\mathbf{A}}(x_{n-1},0).$$

(*vi*) *A subvariety* $\mathcal{U}\subseteq\mathcal{V}$ *satisfies* $s(\bar{x})=t(\bar{x})$ *iff* $\mathcal{U}$ *satisfies* $\hat{s}_i(x,y)=\hat{t}_i(x,y)$ *for each* $i<n$.

(*vii*) *If* $\hat{s}_0=\hat{s}_1\overset{\text{def}}{=}\hat{s}$ *is an equation of* $\mathcal{V}$, *then*

$$\hat{s}\bullet x_0+\hat{s}\bullet x_1+\hat{s}_2\bullet x_2+\cdots+\hat{s}_{n-1}\bullet x_{n-1}=\hat{s}\bullet(x_0+x_1)+\hat{s}_2\bullet x_2+\cdots+\hat{s}_{n-1}\bullet x_{n-1}$$

is an equation of $\mathcal{V}$.

(*viii*) *The coefficient of* x_0 *in* s *equals 0 iff* s *does not depend on* x_0 *in* $\mathcal{V}$.

Proof: Claims (*i*), (*ii*) and (*iii*) follow immediately from the definition of a coefficient. We prove only (*iv*) – (*viii*) since these are not as obvious.

In (*iv*), one merely notes that the coefficient of x in $s(x,x,x_2,\ldots,x_{n-1})$ is by definition $s(x,x,y,y,\ldots,y)+y$. But

$$\begin{aligned}
s(x,x,y,y,\ldots,y)+y &= (s(x,x,y,y,\ldots,y)+s(y,y,y,y,\ldots,y))+y\\
&= s(x+y,x+y,y,y,\ldots,y)+y\\
&= s(x+y,y+x,y,y,\ldots,y)+y\\
&= (s(x,y,y,y,\ldots,y)+s(y,x,y,y,\ldots,y))+y\\
&= (s(x,y,y,y,\ldots,y)+y)+(s(y,x,y,y,\ldots,y)+y)\\
&= \hat{s}_0(x,y)+\hat{s}_1(x,y).
\end{aligned}$$

For (*v*), assume that $\mathbf{A}\in\mathcal{V}$ has a least element 0. Observe that for any $i<n$ we have

$$\hat{s}_i^{\mathbf{A}}(x_i,0)=s^{\mathbf{A}}(0,0,\ldots,x_i,\ldots,0)+0=s^{\mathbf{A}}(0,0,\ldots,x_i,\ldots,0)$$

with x_i in the i^{th} position. Then, since $s^{\mathbf{A}}$ commutes with $x_0+x_1+\cdots+x_{n-1}$ on the array

$$\begin{bmatrix} x_0 & 0 & \cdots & 0\\ 0 & x_1 & & 0\\ \vdots & & \ddots & \\ 0 & 0 & & x_{n-1}\end{bmatrix},$$

we get exactly that $s^{\mathbf{A}}(x_0,\ldots,x_{n-1})=\Sigma_{i<n}\hat{s}_i^{\mathbf{A}}(x_i,0)$.

For (*vi*) assume that $s(\bar{x})=t(\bar{x})$ is an equation of $\mathcal{U}$. Then clearly

$$\hat{s}_0(x,y)=s(x,y,\ldots,y)+y=t(x,y,\ldots,y)+y=\hat{t}_0(x,y)$$

is also an equation of $\mathcal{U}$. Similarly $\hat{s}_i(x,y)=\hat{t}_i(x,y)$ holds for $i>0$. Now suppose that $s(\bar{x})=t(\bar{x})$ is not an equation of $\mathcal{U}$. Then we can find subdirectly irreducible mode $\mathbf{A}\in\mathcal{U}$ and a tuple $\bar{a}\in A^n$ such that $s^{\mathbf{A}}(\bar{a})\neq t^{\mathbf{A}}(\bar{a})$. Now using (*v*) we have that

$$s^{\mathbf{A}}(a_0,\ldots,a_{n-1})=\hat{s}_0^{\mathbf{A}}(a_0,0)+\cdots+\hat{s}_{n-1}^{\mathbf{A}}(a_{n-1},0)$$

with a similar expansion for $t^{\mathbf{A}}(\bar{a})$. Clearly, we cannot have $\hat{s}_i^{\mathbf{A}}(a_i, 0) = \hat{t}_i^{\mathbf{A}}(a_i, 0)$ for all i if $s^{\mathbf{A}}(\bar{a}) \neq t^{\mathbf{A}}(\bar{a})$. So, for some i, we have $\hat{s}_i^{\mathbf{A}}(a_i, 0) \neq \hat{t}_i^{\mathbf{A}}(a_i, 0)$. This witnesses the failure of $\hat{s}_i(x, y) = \hat{t}_i(x, y)$ in $\mathcal{U}$.

In (vii) we must show that if $\tilde{s} \stackrel{\text{def}}{=} s(x, x, x_2, \ldots, x_{n-1})$, then

$$s(x_0, x_1, x_2, \ldots, x_{n-1}) = \tilde{s}(x_0 + x_1, x_2, \ldots, x_{n-1})$$

is an equation of $\mathcal{V}$ whenever $\hat{s}_0 = \hat{s}_1$ is. To see this, note that

$$\begin{aligned}
\tilde{s}(x_0 + x_1, x_2, \ldots, x_{n-1}) &= s(x_0 + x_1, x_0 + x_1, x_2, \ldots, x_{n-1}) \\
&= s(x_0 + x_1, x_1 + x_0, x_2, \ldots, x_{n-1}) \\
&= s(x_0, x_1, x_2, \ldots, x_{n-1}) + s(x_1, x_0, x_2, \ldots, x_{n-1}) \\
&= s(x_0, x_1, x_2, \ldots, x_{n-1}).
\end{aligned}$$

The only non-obvious step is the last one where we use (i) and (vi) to deduce that $s(x_0, x_1, x_2, \ldots, x_{n-1}) = s(x_1, x_0, x_2, \ldots, x_{n-1})$ is an equation of $\mathcal{V}$ (since these terms have equal coefficients).

Finally we prove $(viii)$. If $s(x_0, \ldots, x_{n-1})$ does not depend on x_0, then

$$\hat{s}_0(x, y) = s(x, y, y, \ldots, y) + y = s(y, y, y, \ldots, y) + y = y.$$

Hence the coefficient of x_0 equals 0. Next, suppose that $s(x_0, \ldots, x_{n-1})$ does depend on x_0 in some member of $\mathcal{V}$. Then we can find a subdirectly irreducible $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, $\bar{c} \in A^{n-1}$ such that $s^{\mathbf{A}}(a, \bar{c}) \neq s^{\mathbf{A}}(b, \bar{c})$. But now using (v) it is clear that $\hat{s}_0^{\mathbf{A}}(a, 0) \neq \hat{s}_0^{\mathbf{A}}(b, 0)$, so it is impossible for $\hat{s}(x, y) = y$ to be an equation of $\mathcal{V}$. $\Box$

THEOREM 3.6 *If $\mathcal{V}$ is a variety of semilattice modes, then any subvariety $\mathcal{U} \subseteq \mathcal{V}$ is axiomatized relative to $\mathcal{V}$ by the set of all equations $s(x, y) = t(x, y)$ satisfied by $\mathcal{U}$ where $s(x, y) \geq y$ and $t(x, y) \geq y$ are equations of $\mathcal{V}$. In particular, every variety of semilattice modes is axiomatized by the entropic laws and binary equations.*

Proof: According to Lemma 3.5 (vi), any set of equations defining $\mathcal{U}$ relative to $\mathcal{V}$ can be replaced be equations involving coefficients. This proves the first statement of the theorem.

For the second statement, suppose that $\mathcal{V}$ is a variety of semilattice modes and that $\mathcal{W}$ is the supervariety defined by the entropic laws and the binary equations satisfied by $\mathcal{V}$. $\mathcal{W}$ is a variety of modes since it is entropic and idempotent (the idempotent laws may be written as binary equations). If $b(x, y) = xy$ is a binary term that interprets in $\mathcal{V}$ as a semilattice operation, then it also interprets in $\mathcal{W}$ as a semilattice operation. This is because the commutative law, $xy = yx$, is a binary equation and the associative law, $x(yz) = (xy)z$, is a consequence of the entropic laws and binary equations satisfied by $\mathcal{V}$:

$$\begin{aligned}
x(yz) &= (xx)(yz) \\
&= (xy)(xz) \\
&= [(xy)(xy)][xz] \\
&= [(xy)x][(xy)z] \\
&= [xy][(xy)z] \\
&= (xy)z.
\end{aligned}$$

This derivation required the equation $(xy)x = xy$ to go from line 4 to line 5 and the equation $u(uz) = uz$ to go from line 5 to line 6. These are binary equations satisfied by $\mathcal{V}$, so they are satisfied by $\mathcal{W}$. We get that $\mathcal{W}$ is a variety of semilattice modes containing $\mathcal{V}$ and satisfying all the binary equations satisfied by $\mathcal{V}$. By the first part of the theorem, $\mathcal{W} = \mathcal{V}$. $\square$

(Incidentally, every variety of Mal'cev modes is also axiomatized by the entropic laws + some binary equations.)

COROLLARY 3.7 *If $\mathcal{V}$ is a variety of semilattice modes, then $\mathcal{V} = \mathsf{V}(\mathbf{F}_{\mathcal{V}}(2))$.* $\square$

From Theorem 3.3 we know the structure of any subdirectly irreducible semilattice mode. Using this characterization we can estimate the size of these subdirectly irreducibles. The following theorem proves that every variety of semilattice modes is residually small.

THEOREM 3.8 *Let $\mathcal{V}$ be a variety of modes and set $\lambda = |F_{\mathcal{V}}(x, y)|$. If $\mathbf{A} \in \mathcal{V}$ is a subdirectly irreducible semilattice mode, then $|A| \leq 2^{\lambda}$. If λ is finite, then $|A| \leq \frac{1}{2}(\lambda + 1)$ with equality holding if $\mathcal{V} = \mathsf{V}(\mathbf{A})$.*

Proof: Without loss of generality we may assume that $\mathcal{V} = \mathsf{V}(\mathbf{A})$. Now we must prove that $|A| \leq 2^{\lambda}$ and that $|A| = \frac{1}{2}(\lambda+1)$ if λ is finite. We may further assume that $\mathbf{A} = \mathbf{S}(U)$ where $\mathbf{S}$ is the semilattice reduct of $\mathbf{A}$ and $U = \{s^{\mathbf{A}}(x, 0) |\, s \text{ is a binary term}\}$.

Since $\mathbf{A}$ is subdirectly irreducible, whenever $a, b \in A$ and $a \neq b$ we can find a polynomial $p(x) \in \mathrm{Pol}_1 \mathbf{A}$ such that $p(a) = 0 \neq p(b)$ or $p(b) = 0 \neq p(a)$. The range of p contains 0, so $p(x) \in U$. Let $\phi : A \to \{0, 1\}$ be defined as follows: $\phi(0) = 0$ and, for all $x \neq 0$, $\phi(x) = 1$. Now we define a function

$$\Phi : A \to 2^{|U|} : x \mapsto (\phi \circ p(x))_{p \in U}.$$

From the first sentence of this paragraph and the definition of Φ, we get that Φ is a 1-1 function. Hence $|A| \leq 2^{|U|}$. Now the function

$$\mathbf{F}_{\mathcal{V}}(x, y) \longrightarrow U : s(x, y) \mapsto s^{\mathbf{A}}(x, 0)$$

is well-defined and onto, so $|F_{\mathcal{V}}(x, y)| = \lambda \geq |U|$. Thus, $|A| \leq 2^{\lambda}$.

Now we refine our estimate of $|A|$ in the case when λ is finite. In this case, we still have that $|A| \leq 2^{\lambda}$, so $\mathbf{A}$ is finite also. For each $p \in U$ the set $p^{-1}(0)$ is a non-empty principal ideal in $\mathbf{S}$, the semilattice reduct of $\mathbf{A}$. This is because each $p \in U$ is an endomorphism of $\mathbf{S}$ which has 0 in its range. Let $a_p \in A$ denote the largest member of this ideal. That is, $a_p = \Sigma_{x \in p^{-1}(0)} x$.

Claim The map $\Psi : U \to A : p \mapsto a_p$ is a bijection.

Proof of Claim: First notice that

$$a_{p+q} = a_p \wedge a_q.$$

What this expression means is that if $p(x), q(x) \in U$, then $p(x) + q(x) \in U$. Further, $(p + q)^{-1}(0) = p^{-1}(0) \cap q^{-1}(0)$, so a_{p+q} is the greatest lower bound of a_p and a_q in the (join-)semilattice ordering on $\mathbf{A}$. Notice also that the constant polynomial $r(x) = 0$ is a member of U and that $\Psi(r) = a_r = \Sigma_{x \in A} x$ is the largest element in the semilattice ordering of $\mathbf{A}$. Thus, $\Psi(U) \subseteq A$ is closed under $\wedge$ and contains the largest element of $\mathbf{A}$.

Now, assume that $\Psi : U \to A$ is not onto. Choose an element $b \in A - \Psi(U)$ which is maximal in the semilattice ordering. Since $\Psi(U)$ is closed under $\wedge$ and contains the top element of $\mathbf{A}$, this b is a meet-irreducible element of $\mathbf{A}$ which is not the top element. Let b^* denote the unique upper cover of such a b. Suppose that $p \in \mathrm{Pol}_1\mathbf{A}$ and $p(b) = 0$. Of course, $p \in U$ since 0 is in the range of p. Since $b \neq a_p$, we get that $b < a_p$. In particular, $b^* \leq a_p$, so $p(b^*) = 0$. Conversely, if $p \in \mathrm{Pol}_1\mathbf{A}$ and $p(b^*) = 0$, then $p(b) = 0$ since $b < b^*$. We have shown that

$$(\forall p \in \mathrm{Pol}_1\mathbf{A}) p(b) = 0 \longleftrightarrow p(b^*) = 0.$$

From this it follows that $\mathrm{Cg}(b, b^*)$-class of 0 is trivial. But this is impossible since the monolith of $\mathbf{A}$ is $\mu = \mathrm{Cg}(0, u) \leq \mathrm{Cg}(b, b^*)$. This contradiction proves that $\Psi : U \to A$ is onto.

Now we prove that $\Psi : U \to A : p \mapsto a_p$ is 1-1. Assume instead that $p, q \in U$, $p \neq q$ and $a_p = a_q$. The last condition implies that $p(x) = 0 \leftrightarrow q(x) = 0$. Since $p \neq q$, there is some $c \in A$ such that $p(c) \neq q(c)$. The monolith of $\mathbf{A}$ is $\mu = \mathrm{Cg}(0, u)$, so $(0, u) \in \mathrm{Cg}(p(c), q(c))$. Hence there is an $h \in \mathrm{Pol}_1\mathbf{A}$ such that, say, $0 = h(p(c)) < h(q(c))$. As 0 is in the range of h, $h \in U$. The members of U commute, so $0 = p(h(c)) < q(h(c))$. But this is impossible since, for $x = h(c)$, this contradicts $p(x) = 0 \leftrightarrow q(x) = 0$. The claim is established.

Since $\Psi : U \to A$ is a bijection, we conclude that $|A| = |U|$. How big is $|U|$? The description of the binary terms up to $\mathsf{V}(\mathbf{A})$-equivalence that we gave in Theorem 3.4 shows that all binary terms of $\mathsf{V}(\mathbf{A})$ are of the form $f(x) + y$ or $x + f(y)$ where $f(x) \in U$ and that no two terms of this form are $\mathsf{V}(\mathbf{A})$-equivalent except that trivially $\mathbf{A} \models id_A(x) + y = x + y = x + id_A(y)$. Thus if $B_x = \{f(x) + y | f \in U\}$ and $B_y = \{x + f(y) | f \in U\}$, then $B_x \cup B_y$ is a set of binary terms which represent all binary terms. Up to equivalence in $\mathsf{V}(\mathbf{A})$, the only term with a multiple representation is $x + y \in B_x \cap B_y$. Since $|B_x| = |B_y| = |U|$, we get that $2|U| - 1 = |F_{\mathcal{V}}(x, y)| = \lambda$. Hence $\frac{1}{2}(\lambda + 1) = |U| = |A|$. $\square$

We showed that the map $\Psi : U \to A$ is 1-1 by proving that if $p, q \in U$ and $a_p = a_q$, then $p(x) = q(x)$. This argument does not require that $q \in U$, only that q is an endomorphism of $\mathbf{A}$ and that $q(0) = 0$. But when $\mathbf{A}$ is finite, $\Psi : U \to A : p \mapsto a_p$ is onto. Hence if q is an endomorphism of $\mathbf{A}$, $q(0) = 0$ and a_q is defined to be the greatest $y \in A$ for which $q(y) = 0$, then $a_q = a_p$ for some $p \in U$. Our argument proves that $q(x) = p(x)$ for this p. We conclude that the set, $End^0(\mathbf{A})$, of mode endomorphisms which map 0 to 0 is exactly the set U. An immediate corollary of this observation is the finite case of the next result. The reader should have little trouble supplying the argument for the general case. (It is based on the proof of Theorem 3.2.) We shall not need the result, so we omit the argument.

PROPOSITION 3.9 *A subdirectly irreducible semilattice mode has a trivial automorphism group.* $\square$

4 Semilattice Modes as Semimodules

We define a **semiring** to be an algebra $\mathbf{R} = \langle R; \cdot, +, 1, 0\rangle$ of type $\langle 2, 2, 0, 0\rangle$ in which the following laws hold:

Associative Laws:	$\begin{cases} x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\ x + (y + z) &= (x + y) + z \end{cases}$
Commutative Law for $+$:	$x + y = y + x$
Unit Laws:	$\begin{cases} 1 \cdot x &= x \cdot 1 = x \\ 0 + x &= x + 0 = x \end{cases}$
Absorptive Law for 0:	$0 \cdot x = x \cdot 0 = 0$
Distributive Laws:	$\begin{cases} x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ (x + y) \cdot z &= (x \cdot z) + (y \cdot z) \end{cases}$

(The axioms for semirings differ from author to author. Usually a semiring is defined with at least the following properties: a semiring has two binary operations corresponding to multiplication and addition, both of these operations are associative, and multiplication distributes over addition on both sides. See [3] or [10] for more general semirings than those defined above. The semirings that we will encounter are much more "ring-like" in that they satisfy all the usual defining axioms for rings which don't refer to negation.)

A **semimodule M** over the semiring **R** is an algebra of the form $\langle M; +, 0, \lambda_r(r \in R)\rangle$ of type $\langle 2, 0, 1, 1, 1, \ldots\rangle$ defined by the laws

Associative Law:	$x + (y + z) = (x + y) + z$
Commutative Law:	$x + y = y + x$
Unit Law:	$0 + x = x + 0 = x$
Laws of Linearity:	$\begin{cases} \lambda_r(0) &= 0 \\ \lambda_r(x + y) &= \lambda_r(x) + \lambda_r(y) \end{cases}$
Action Laws:	$\begin{cases} \lambda_0(x) &= 0 \\ \lambda_1(x) &= x \\ \lambda_{r+s}(x) &= \lambda_r(x) + \lambda_s(x) \\ \lambda_{r \cdot s}(x) &= \lambda_r(\lambda_s(x)) \end{cases}$

As is common when working with rings and modules, we will abbreviate $r \cdot s$ by rs and $\lambda_r(x)$ by $r(x)$ when no confusion is likely.

Example 3. If **S** is a ring and **N** is a left **S**-module, then let **R** be the reduct of **S** obtained by ignoring minus (-) and let **M** be the reduct of **N** obtained by ignoring minus. **R** is a semiring and **M** is an **R**-semimodule. These are perhaps the most obvious examples of semirings and semimodules.

Example 4. Let **A** be a subdirectly irreducible semilattice mode. Let

$$\mathbf{R} = \mathbf{End}^0(\mathbf{A}) = \langle U; \circ, +, id_A, 0\rangle$$

and

$$\mathbf{M} = \langle A; +, 0, p(x)(p \in U)\rangle.$$

$\mathbf{M}$ is an $\mathbf{R}$-semimodule and $\mathbf{R}$ is a commutative semiring satisfying the law $\forall r(1+r = 1)$. This law is satisfied because each $r \in R$ is a decreasing endomorphism of $\langle A; + \rangle$, so for all $x \in A$ we have

$$(1+r)(x) = id_A(x) + r(x) = x + r(x) = x = id_A(x) = 1(x).$$

Example 5. Throughout this section we are going to be interested in commutative semirings satisfying $1+r = 1$. The best-known example of such a semiring is a bounded distributive lattice. If $\mathbf{D} = \langle D; \cdot, +, 1, 0 \rangle$ is a bounded distributive lattice with $\cdot =$ meet and $+ =$ join, then $\mathbf{D}$ satisfies all the semiring laws. Furthermore, the "multiplication" or meet operation is commutative and $1 + r = 1$, since $+ =$ join and 1 is the top element. Conversely, a commutative semiring satisfying $1+r = 1$ satisfies all the usual defining laws of a bounded distributive lattice except idempotence of multiplication, $r \cdot r = r$, and $(\cdot, +)$-absorption, $r \cdot (r + s) = r$. (Incidentally, these are equivalent laws in the presence of the semiring laws and $1 + r = 1$. If $r \cdot r = r$ for all r, then

$$r(r + s) = r^2 + rs = r + rs = r(1 + s) = r1 = r$$

for all r and s. Conversely, if $r(r + s) = r$ for all r and s, then by taking $s = 0$ we get $r^2 = r$ for all r.) Hence, the variety of bounded distributive lattices equals the variety of commutative semirings satisfying $1 + r = 1$ and $r^2 = r$.

In this section we examine the underlying semilattice structure of subdirectly irreducible semimodules in varieties of semimodules over commutative semirings satisfying the law $1 + r = 1$. The law $1 + r = 1$ implies that semiring addition is idempotent:

$$r + r = r \cdot (1 + 1) = r \cdot 1 = r.$$

From this it follows that addition in any $\mathbf{R}$-semimodule is idempotent:

$$x + x = \lambda_1(x) + \lambda_1(x) = \lambda_{1+1}(x) = \lambda_1(x) = x.$$

Thus, when $\mathbf{R} \models 1 + r = 1$, addition is a semilattice operation for $\mathbf{R}$ and also for any $\mathbf{R}$-semimodule. It is this semilattice ordering that we refer to whenever we talk about the ordering of a semiring or semimodule. The semimodule law $0+x = x+0 = x$ implies that 0 is the least element with respect to the semilattice ordering. Thus, the semimodules we consider in this section are join-semilattices with a least element 0 which have endomorphisms adjoined. The law $1+r = 1$ implies that, for all x, $x+r(x) = x$ or that $r(x) \leq x$. This means that a semiring element acts on a semimodule as a decreasing join-endomorphism. Since in this section we are looking at semimodules over a commutative semiring satisfying $1 + r = 1$, the objects we are considering are just lower bounded, join-semilattices with commuting, decreasing endomorphisms adjoined. By Theorem 3.1, every subdirectly irreducible semilattice mode is polynomially equivalent to such an algebra. The reason for changing terminology in this section is that the arguments we use in this section are very similar to well-known arguments for modules.

One of our goals is to prove that a variety of semilattice modes of finite type is residually countable. Since $\lambda = |F_{\mathcal{V}}(x, y)| \leq \omega$ when $\mathcal{V}$ is of finite type, the results of the last section prove that a variety of semilattice modes of finite type is residually $\leq 2^\omega$. We improve that estimate here. A similar improvement on residual bounds for varieties of modules is known. For example, for any ring **R** the variety of left **R**-modules is residually $\leq 2^{|R|}$. But if **R** is left Noetherian, then the variety of left **R**-modules is residually $\leq |R|$, see [4]. Now any finitely generated, commutative ring is (left) Noetherian by the Hilbert Basis Theorem and is also countable. The consequence: every variety of modules over a finitely generated, commutative ring is residually countable. We shall use analogous arguments to prove that every variety of semimodules over a finitely generated, commutative semiring satisfying $1 + r = 1$ is residually countable.

4.1 Commutative Semirings Satisfying $1 + r = 1$

The purpose of this section is to prove an analogue of the Hilbert Basis Theorem for commutative semirings satisfying $1 + r = 1$. From this we deduce that any subdirectly irreducible semimodule over a finitely generated, commutative semiring satisfying $1 + r = 1$ is countable.

Definition 4.1 An **annihilator ideal** of a commutative semiring **R** satisfying $1+r = 1$ is a subset $S \subseteq R$ such that

- (i) $0 \in S$.
- (ii) If $r, s \in S$, then $r + s \in S$.
- (iii) If $r \in R$ and $s \in S$, then $rs \in S$.
- (iv) If $r \in S$ and $s \leq r$, then $s \in S$.

(Note that any nonempty subset $S \subseteq R$ which satisfies (ii) and (iv) also satisfies (i) and (iii).)

If I is an annihilator ideal of a commutative semiring **R** satisfying $1 + r = 1$, then the congruence $\mathrm{Cg}^{\mathbf{R}}(I \times I)$ is called the **ideal congruence** associated with I and it is denoted $\Theta(I)$. This congruence is just

$$\Theta(I) = \{(x, y) \in R^2 \mid x + i = y + i \text{ for some } \ i \in I\}.$$

It is easy to see that I is the $\Theta(I)$-class containing 0 and conversely that any $0/\theta$, $\theta \in \mathrm{Con}\,\mathbf{R}$, is an annihilator ideal. We shall call a commutative semiring **Noetherian** if it satisfies the ascending chain condition on annihilator ideals.

LEMMA 4.2 *If* **R** *is a commutative semiring satisfying* $1 + r = 1$*, then the following are equivalent.*

- (i) **R** *is Noetherian.*
- (ii) *Every annihilator ideal is finitely generated.*
- (iii) *Every annihilator ideal is principal.*
- (iv) $\langle R; \leq\rangle$ *satisfies the ascending chain condition.*

Proof: The equivalence of (i) and (ii) follows from the well-known result that if c is a closure operator on the set X, then the lattice of closed subsets satisfies the ascending chain condition if and only if every closed set is the closure of one of its finite subsets. Here X is taken to be R and c is the operator of ideal generation in $\mathbf{R}$.

For $(ii) \to (iii)$, observe that the annihilator ideal generated by $\{a_0, \ldots, a_n\}$ equals the annihilator ideal generated by $a = a_0 + \cdots + a_n$. The implication $(iii) \to (ii)$ is trivial.

Given (i) we get that (i) and (iii) both hold. Now (iv) follows immediately, since every element of R may be identified with the annihilator ideal it generates. Thus $(i) \to (iv)$. To finish we need only show that $(iv) \to (iii)$. If (iv) holds and $S \subseteq R$ is an annihilator ideal, then $S \neq \emptyset$. Since $\langle R, \leq \rangle$ satisfies the ascending chain condition, every element of S is below a maximal element of S. If $r, s \in S$ are distinct maximal elements of S, then $r + s \in S$ and $r \leq r + s$, $s \leq r + s$. It follows that $r = r + s = s =$ the largest element of S. Hence S is principal and (iii) holds. □

The next theorem is the analogue of the Hilbert Basis Theorem for commutative rings.

THEOREM 4.3 *Assume that* $\mathbf{R}$ *is a Noetherian, commutative semiring satisfying the equation* $1 + r = 1$*. Then* $\mathbf{R}[x]$ *is Noetherian.*

Proof: Here $\mathbf{R}[x]$ denotes the commutative semiring satisfying $1 + r = 1$ which is obtained from $\mathbf{R}$ by freely adjoining a new element x. Using the commutativity and associativity of the semiring operations we may write any $p \in R[x]$ as $p = a_0 + a_1 x + \cdots + a_n x^n$ where $a_i \in R$. Repeatedly using the fact that

$$a_i x^i + a_{i+1} x^{i+1} = a_i x^i (1 + x) + a_{i+1} x^{i+1} = a_i x^i + (a_i + a_{i+1}) x^{i+1},$$

we find that we may write

$$p = a_0 + (a_0 + a_1) x + \cdots + (a_0 + a_1 + \cdots + a_n) x^n.$$

Hence we may express any $p \in R[x]$ as $p = b_0 + b_1 x + \cdots + b_n x^n$ where $b_0 \leq b_1 \leq \cdots \leq b_n$. If $b_{n-1} = b_n$, then we may reduce further since

$$\begin{aligned} b_0 + \cdots + b_{n-1} x^{n-1} + b_n x^n &= b_0 + \cdots + b_{n-1} x^{n-1} + b_{n-1} x^n \\ &= b_0 + \cdots + b_{n-1} x^{n-1} (1 + x) \\ &= b_0 + \cdots + b_{n-1} x^{n-1}. \end{aligned}$$

We see that any $p \in R[x]$ different from 0 may be expressed as $p = b_0 + b_1 x + \cdots + b_k x^k$ where $b_0 \leq \cdots \leq b_{k-1} < b_k$. (We mandate that the expression $p = b_0$, $b_0 \in R$, is of this form.) The process we have just described is a process that reduces each $a_0 + \cdots + a_n x^n$ into a normal form $b_0 + \cdots + b_k x^k$, $b_0 \leq \cdots \leq b_{k-1} < b_k$. If $p = a_0 + \cdots + a_n x^n$, write $\mathrm{red}(p)$ for the reduced expression $b_0 + \cdots + b_k x^k$, $b_0 \leq \cdots \leq b_{k-1} < b_k$. We have argued that p and $\mathrm{red}(p)$ are different expressions for the same element. To verify that we actually have a normal form for elements, one must show that the collection of reduced expressions forms a commutative semiring satisfying $1 + r = 1$. (The operations on reduced expressions are $\mathrm{red}(p) \cdot \mathrm{red}(q) \stackrel{\text{def}}{=} \mathrm{red}(\mathrm{red}(p) \cdot \mathrm{red}(q))$, $\mathrm{red}(p) + \mathrm{red}(q) \stackrel{\text{def}}{=}$

red(red(p) + red(q)), $1 \stackrel{\text{def}}{=} 1$, $0 \stackrel{\text{def}}{=} 0$.) The argument for this is not hard, but checking all the semiring laws is a bit long, so we omit it. (The reader uncomfortable with this omission will find that nothing in the rest of the paper depends on Theorem 4.3, only on its consequence: Corollary 4.4. But Corollary 4.4 follows immediately from Lemma 4.2 and Theorem 4.9 and these results are proved independently of Theorem 4.3, so all future arguments are complete.)

From this point on we will work only with reduced expressions for elements of $\mathbf{R}[x]$. For such elements we have the following **comparability test** which follows immediately from the normal form for elements. If $p = b_0 + b_1x + \cdots + b_nx^n$ and $q = c_0 + c_1x + \cdots + c_kx^k$ are reduced, then $p \leq q$ (meaning $p + q = q$) iff $b_i \leq c_i$ for $i \leq k$ and $b_j \leq c_k$ for $j > k$.

Now let $J \subseteq R[x]$ be a nonzero annihilator ideal. We will use the fact that $\mathbf{R}$ is Noetherian to construct an element $q \in R[x]$ such that J is the ideal generated by q. According to Lemma 4.2 this will finish the proof of the theorem.

Let I_0 denote the set of elements $b \in R$ such that b is the constant coefficient (the coefficient of x^0) in the reduced expression of some element of J. That is, if $p = b_0 + b_1x + \cdots + b_nx^n$ is reduced, then $b_0 \in I_0$ and conversely if $b \in I_0$, then there is a $p \in J$ such that $p = b + b_1x + \cdots + b_nx^n$ is reduced. In this definition we get that $0 \in I_0$ since 0 is the coefficient of x^0 in the reduced expression for $0 \in J$. I_0 is closed under $+$ since J is. To show that I_0 is an annihilator ideal we must show that if $b \in I_0$ and $b' \leq b$, then $b' \in I_0$. Since $b \in I_0$, there is a $p = b + \cdots + b_kx^k \in J$. Now $p' = b' + \cdots + b_kx^k$ is reduced and $p' \in J$ since $p' \leq p$, so $b' \in I_0$.

In a similar way define I_k for each $k > 0$ to be the set of all elements $b \in R$ which occur as a coefficient of x^l, some $l \leq k$, in some $p \in J$ where $p = b_0 + b_1x + \cdots + b_nx^n$ is reduced. Each of the sets I_k is an annihilator ideal of $\mathbf{R}$ and of course we have

$$I_0 \subseteq I_1 \subseteq \cdots$$

Since $\mathbf{R}$ is Noetherian, each of these ideals is generated by a single element. Choose c_i so that I_i is the annihilator ideal generated by c_i. We have $c_0 \leq c_1 \leq \cdots$. Since $\langle R, \leq \rangle$ satisfies the ascending chain condition this sequence is eventually constant. It follows that there is a $k \geq 0$ which is minimal for the property that $c_k =$ this constant value. That is,

$$c_0 \leq c_1 \leq \cdots \leq c_{k-1} < c_k = c_{k+1} = \cdots .$$

We claim that J is the ideal generated by the single element $q = c_0 + \cdots + c_{k-1}x^{k-1} + c_kx^k$. We must show that $q \in J$ and $p \leq q$ for all $p \in J$. We'll argue the latter first. Choose any $p \in J$. Say $p = b_0 + \cdots + b_nx^n$. We must have $b_i \leq c_i$ for $i \leq k$ since $b_i \in I_i$ and c_i is the largest element of I_i. We must have $b_j \leq c_k$ for $j > k$ since $b_j \in I_j$ and c_k is the largest element of $I_k = I_j$. By the comparability test we deduce that $p \leq q$ for any $p \in J$. To deduce that $q \in J$ it suffices to show that for each i we have $c_ix^i \in J$ since J is closed under $+$. Since $c_i \in I_i$, there is an element $s_i \in J$ such that c_i is the coefficient in s_i of some x_l, $l \leq i$. Using the comparability test we find that $c_ix^i \leq s_i$. Hence $c_ix^i \in J$ since J is an annihilator ideal. This concludes the proof that J is principal. As J was an arbitrarily chosen annihilator ideal, we get that $\mathbf{R}[x]$ is Noetherian. $\square$

COROLLARY 4.4 *Any finitely generated, commutative semiring satisfying* $1+r = 1$ *is Noetherian.*

Proof: Without loss of generality, we may assume that $\mathbf{R}$ is the n-generated, free, commutative semiring satisfying $1 + r = 1$ for some $n < \omega$. For if $\mathbf{R}'$ is any other n-generated commutative semiring satisfying $1 + r = 1$, there is an onto homomorphism $\varphi : \mathbf{R} \to \mathbf{R}'$. If

$$I_0 \subset I_1 \subset \cdots$$

is an infinite, strictly increasing chain of annihilator ideals in $\mathbf{R}'$, then

$$\varphi^{-1}(I_0) \subset \varphi^{-1}(I_1) \subset \cdots$$

is an infinite, strictly increasing chain of annihilator ideals in $\mathbf{R}$ as one sees from the definition of annihilator ideal.

Now, an inductive proof based on Theorem 4.3 can be completed if we show that the 0-generated, free, commutative semiring satisfying $1 + r = 1$ is Noetherian. But this semiring has only two elements, 0 and 1, so the proof is complete. $\Box$

THEOREM 4.5 *Let* $\mathbf{M}$ *be a subdirectly irreducible semimodule over a commutative, Noetherian semiring* $\mathbf{R}$ *which satisfies* $1 + r = 1$. *Then* $|M| \leq |R|$.

Proof: We leave the case when $|R|$ is finite to the reader - the result in this case is not used anywhere in this paper. This case may immediately be reduced to the situation where $\mathbf{R}$ acts faithfully on $\mathbf{M}$. Then the idea is to replace $\mathbf{M}$ with the polynomially equivalent mode $\mathbf{S}(U)$ where $\mathbf{S} = \langle M; +\rangle$ and $U = R$. From Theorem 3.8 we get that $|M| \leq \frac{1}{2}(\lambda + 1) = |U| = |R|$.

Now we assume that $|R|$ is infinite and $|M| > |R|$. We shall prove that $\mathbf{R}$ is not Noetherian by showing how to construct an ascending infinite chain of annihilator ideals in $\mathbf{R}$. $\mathbf{M}$ is polynomially equivalent to $\mathbf{S}(U)$ where $\mathbf{S} = \langle M; +\rangle$ and $U = R$. By Theorem 3.1, this means that $\mathbf{M}$ has an element 0 which is the least element in the semilattice order and an element u which is the least element in $M - \{0\}$. Further, the monolith of $\mathbf{M}$ is the equivalence relation on M generated by the pair $(0, u)$. For each element $a \in M$ which is not the greatest element in the semilattice order we can choose $a' > a$. Since $(0, u) \in \mathrm{Cg}(a, a')$ it follows that there is an $r \in R$ such that $r(a') > r(a) = 0$. Therefore, to each $a \in M$ which is not the largest element we can assign an $r \in R$ such that $r(a) = 0$ and $r(M) \neq \{0\}$. This assignment is a map from M to R or from $M - \{T\}$ to R if M has a top element T. Since $|M| > |R| \geq \omega$, we may find $M' \subseteq M$ such that $|M'| > |R|$ and all members of M' are assigned the same element $r \in R$. Denote by M_r the subuniverse $r^{-1}(0)$. As $M' \subseteq M_r$, we get that $|M_r| > |R|$. Let I_r denote the annihilator ideal $\{t \in R | \, t(M_r) = \{0\}\}$.

We repeat our above argument with M_r in place of M. To each element $b \in M_r$ which is not the greatest element of M_r assign an element $s \in R$ such that $s(b) = 0$, but $s(M_r) \neq \{0\}$. As $|M_r| > |R|$ we can find a subset $M_r' \subseteq M_r$ such that $|M_r'| > |R|$ and all members of M_r' are assigned the same element $s \in R$. Let $M_{r,s} = (r + s)^{-1}(0)$. Note that

$$(r + s)(x) = 0 \leftrightarrow s(x) = 0 \quad \text{and} \quad x \in M_r.$$

Thus, $M_r' \subseteq M_{r,s} \subseteq M_r$. Further,

$$s(M_r) \neq \{0\} = s(M_{r,s}),$$

so $M_{r,s} \subset M_r$ but $|M_{r,s}| > |R|$ still. If $I_{r,s} = \{t \in R|\, t(M_{r,s}) = \{0\}\}$, then $I_{r,s}$ is an annihilator ideal containing I_r and $s \in I_{r,s} - I_r$. Thus, $I_r \subset I_{r,s}$.

We may now continue with $M_{r,s}$ in place of M_r and produce $M_{r,s,t} \subset M_{r,s}$ with $|M_{r,s,t}| > |R|$ and the corresponding annihilator ideal $I_{r,s,t} \supseteq I_{r,s}$ where $t \in I_{r,s,t} - I_{r,s}$. In this fashion we construct a strictly increasing chain of annihilator ideals

$$I_r \subset I_{r,s} \subset I_{r,s,t} \subset \cdots .$$

This proves that $\mathbf{R}$ is not Noetherian. □

COROLLARY 4.6 *If* $\mathbf{R}$ *is a finitely generated, commutative semiring satisfying the equation* $1 + r = 1$*, then the variety of* $\mathbf{R}$*-semimodules is residually countable.*

Proof: If $\mathbf{R}$ is finitely generated, it is countable. Hence the result follows from Corollary 4.4 and Theorem 4.5. □

We have shown that if $\mathbf{R}$ is a finitely generated, commutative semiring satisfying $1 + r = 1$, then $\langle R; \leq\rangle$ satisfies the ascending chain condition. This was all we needed to know about $\mathbf{R}$ in order to prove Corollary 4.6. But with a little more effort we can uncover even more about the order-structure of finitely generated, commutative semirings satisfying $1 + r = 1$.

We begin with a description of the free, n-generated, commutative semiring satisfying $1 + r = 1$. This description generalizes the well-known description of the free, n-generated, bounded distributive lattice as the lattice of antichains in the poset 2^n. By identifying an antichain with the order-filter it generates, the lattice operations may be described simply as intersection and union of filters. Instead of working with antichains, we shall work with the filters they generate so that the semiring operations are easier to describe.

If $\mathbf{P}$ is a poset, then we will write $\mathcal{F}(\mathbf{P})$ for the set of order-filters of $\mathbf{P}$. Let ω^n denote the product of n copies of $\langle \omega; \leq\rangle$ under the product ordering. If $G, H \in \mathcal{F}(\omega^n)$, we define $G + H = G \cup H$ and

$$G \cdot H = \{\bar{x} + \bar{y} \,|\, \bar{x} \in G,\ \bar{y} \in H\}.$$

Here we define $(x_0, x_1, \ldots, x_{n-1}) + (y_0, y_1, \ldots, y_{n-1}) = (x_0 + y_0, x_1 + y_1, \ldots, x_{n-1} + y_{n-1})$ where the coordinatewise $+$ is ordinal addition. Now, if F is the underlying set of $\mathcal{F}(\omega^n)$, then the free commutative semiring satisfying $1 + r = 1$ generated by X is isomorphic to $\langle F; \cdot, +, \omega^n, \emptyset\rangle$. The inclusion of generators is given by $x_i \mapsto F_i$ where F_i is the principal filter determined by the tuple $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, 1 in the i^{th} position. These facts are proved in Theorem 4.8.

Our description can easily be extended to give the free, commutative semiring satisfying $1 + r = 1$ generated by an infinite set X. (Although below we only include the proof for $X = n$.) Simply take F to be the underlying set of $\mathcal{F}_{f.g.}(\bigoplus_X \omega)$, where $\mathcal{F}_{f.g.}(\mathbf{P})$ denotes the set of finitely generated order-filters of $\mathbf{P}$. The poset $\bigoplus_X \omega$ is the poset of functions $f : X \to \omega$ with finite support (i.e., the functions that are 0 almost everywhere) ordered pointwise. Then $\langle F; \cdot, +, \bigoplus_X \omega, \emptyset\rangle$ is isomorphic to the free, commutative semiring satisfying $1 + r = 1$ generated by X.

Before we proceed to our main task, which is to learn more about the semilattice ordering on finitely generated commutative semirings satisfying $1 + r = 1$, we must take a brief detour into the realm of better-quasiorderings (bqos). Since the definition of a bqo is complicated, we omit it and refer the reader to the survey article [9] for this definition and further references. All that we need to know is that a quasiorder is a set with a reflexive, transitive binary relation and that there is a class of quasiorderings with the properties listed in the next theorem.

THEOREM 4.7 *The class of bqos*

(i) *contains all well-ordered sets,*

(ii) *contains all finite quasiorders,*

(iii) *is closed under the formation of finite products (quasiordered by the product quasiordering),*

(iv) *is closed under the formation of subquasiorders (with the induced quasiordering),*

(v) *is closed under the formation of ideal lattices (quasiordered by inclusion).*

Furthermore, if $f : \mathbf{P} \to \mathbf{Q}$ *is a surjective, quasiorder-preserving map from a bqo* $\mathbf{P}$ *onto a quasiordered set* $\mathbf{Q}$*, then* $\mathbf{Q}$ *is a bqo. A partial ordering which is a bqo satisfies the descending chain condition and has no infinite antichains.* □

Thus the class of bqos nicely generalizes finite partial orderings and well-orderings. We will call a quasiordered set a **dual bqo** if its dual is a bqo.

THEOREM 4.8 *The free,* n*-generated, commutative semiring satisfying* $1 + r = 1$ *is isomorphic to* $\langle \mathcal{F}; \cdot, +, \omega^n, \emptyset\rangle$*. Therefore, if* $\mathbf{R}$ *is a finitely generated commutative semiring satisfying* $1 + r = 1$*, then* $\langle R; \leq\rangle$ *is a dual bqo.*

Proof: First we explain why the second statement follows from the first. Any finitely generated commutative semiring satisfying $1 + r = 1$ is the image of an order-preserving map from a free one, so if we show that for any n the free, n-generated, commutative semiring satisfying $1 + r = 1$ is a dual bqo, then the second statement of the theorem will follow. But the order on $\langle \mathcal{F}; \cdot, +, \omega^n, \emptyset\rangle$ is that of a dual bqo by parts (i), (iii) and the dual of (v) in Theorem 4.7. Hence it suffices now to prove the first statement of this theorem.

It is rather easy to verify each of the semiring laws, commutativity and $1 + r = 1$ for $\langle F; \cdot, +, \omega^n, \emptyset\rangle$, so this is in fact a semiring of the desired type. (None of these verifications is harder than this one: If $G, H, K \in \mathcal{F}(\omega^n)$, then $G \cdot (H + K)$ and $G \cdot H + G \cdot K$ both represent the filter of all tuples $\bar{w}$ such that in ω^n we have $\bar{w} \geq \bar{x} + \bar{y}$ for some $\bar{x} \in G$, $\bar{y} \in H$ or we have $\bar{w} \geq \bar{x} + \bar{z}$ for some $\bar{x} \in G$, $\bar{z} \in K$. Hence $G \cdot (H + K) = G \cdot H + G \cdot K$.)

We argue that if F_i denotes the filter generated by $(0, \ldots, 0, 1, 0, \ldots, 0)$, 1 in the i^{th} position, then the set $\{F_i \mid i < n\}$ generates $\langle F; \cdot, +, F, \emptyset\rangle$. Since $\mathcal{F}(\omega^n)$ (under the inclusion ordering) is a dual bqo, it satisfies the ascending chain condition. As noted before (since filter-generation is a closure operator) this implies that each filter is finitely generated. Thus it is a union of principal filters. As $G + H = G \cup H$, it is clear that we need only to prove that the collection of F_i, $i < n$, generate all principal filters. If G is the principal filter generated by $(e_0, \ldots, e_{n-1})$, then our definition of multiplication implies that $G = F_0^{e_0} \cdots F_{n-1}^{e_{n-1}}$. This completes the proof that $\{F_i \mid i < n\}$ is a generating set for $\langle F; \cdot, +, \omega^n, \emptyset\rangle$.

Let $\mathbf{F}_n$ denote the free, commutative semiring satisfying $1+r = 1$ generated by $\mathcal{G} = \{g_0, \ldots, g_{n-1}\}$. Using the commutative, associative and distributive laws we may write each element of F_n as a (possibly empty) sum of monomials in the generators $\mathcal{G}$. By a **monomial** we mean a product of generators. Using the commutative and associative laws for multiplication, each monomial may be put in the form $m = g_0^{e_0} \cdots g_{n-1}^{e_{n-1}}$ where $e_i \in \omega$. The element $1 \in F_n$ is represented by the monomial $g_0^0 \cdots g_{n-1}^0$. The element $0 \in F_n$ is represented by the empty sum of monomials. Define a function γ from ω^n to the set of monomials in the generators $\mathcal{G}$ as follows. For a tuple $t = (e_0, \ldots, e_{n-1}) \in \omega^n$ we let $\gamma(t) = g_0^{e_0} \cdots g_{n-1}^{e_{n-1}}$. Now we define a function

$$\Gamma : \mathcal{F}(\omega^n) \to \mathbf{F}_n.$$

Choose any $G \in \mathcal{F}(\omega^n)$. As noted earlier, G is finitely generated as an order-filter, so assume that $G = \langle t_0, \ldots, t_k\rangle$ where each $t_i \in \omega^n$. (Note that the t_i are uniquely determined by G: the t_i are just the minimal elements of G.) Define

$$\Gamma(G) = \gamma(t_0) + \cdots + \gamma(t_k).$$

Our goal is to show that Γ is a surjective homomorphism whose restriction to $\{F_i \mid i < n\}$ is a bijection from $\{F_i \mid i < n\}$ onto $\mathcal{G}$. The fact that $\mathbf{F}_n$ is free and $\{F_i \mid i < n\}$ generates $\langle F; \cdot, +, F, \emptyset\rangle$ will imply that Γ is an isomorphism.

Clearly $\Gamma(F_i) = \gamma((0, \ldots, 1, \ldots, 0)) = g_0^0 \cdots g_i^1 \cdots g_{n-1}^0 = g_i$. By our definitions and the fact that ω^n is itself the filter generated by $(0, 0, \ldots, 0)$, we get that $\Gamma(\emptyset) = 0$ and $\Gamma(\omega^n) = \gamma((0, \ldots, 0)) = g_0^0 \cdots g_{n-1}^0 = 1$.

To see that Γ is join-preserving we first show that γ is order-inverting. Choose tuples $t = (e_0, \ldots, e_{n-1})$ and $u = (f_0, \ldots, f_{n-1})$. Assume that $t \le u$ (meaning that $e_i \le f_i$ for all i). Then

$$\begin{aligned}
\gamma(t) + \gamma(u) &= g_0^{e_0} \cdots g_{n-1}^{e_{n-1}} + g_0^{f_0} \cdots g_{n-1}^{f_{n-1}} \\
&= g_0^{e_0} \cdots g_{n-1}^{e_{n-1}}(1 + g_0^{(f_0 - e_0)} \cdots g_{n-1}^{(f_{n-1} - e_{n-1})}) \\
&= g_0^{e_0} \cdots g_{n-1}^{e_{n-1}} \\
&= \gamma(t).
\end{aligned}$$

Hence $\gamma(u) \le \gamma(t)$ and γ is order-inverting. To see that Γ is join-preserving, choose filters G and H with $G = \langle u_0, \ldots, u_{k-1}\rangle$ and $H = \langle t_0, \ldots, t_{l-1}\rangle$. If $\{v_0, \ldots, v_{m-1}\}$ is the set of minimal elements of $\{u_0, \ldots, u_{k-1}\} \cup \{t_0, \ldots, t_{l-1}\}$, then $G \cup H$ is the filter generated by the antichain $\langle v_0, \ldots, v_{m-1}\rangle$. Clearly we have

$$\Gamma(G \cup H) = \Sigma\gamma(v_i) \le (\Sigma\gamma(u_i)) + (\Sigma\gamma(t_i)) = \Gamma(G) + \Gamma(H)$$

since the sum on the left side of the inequality is a subsum of the sum on the right hand side. To show equality we need only show that we can delete the extra terms in the sum on the right hand side of the inequality without changing the value of the sum. So choose some element of

$$(\{u_0, \ldots, u_{k-1}\} \cup \{t_0, \ldots, t_{l-1}\}) - \{v_0, \ldots, v_{m-1}\}.$$

Say that u_j is chosen. Then u_j is not minimal in $\{u_0, \ldots, u_{k-1}\} \cup \{t_0, \ldots, t_{l-1}\}$, so (since u_j is incomparable with the other u_i) there is a t_n in this set with $t_n \leq u_j$. As argued above, this inequality implies that $\gamma(u_j) + \gamma(t_n) = \gamma(t_n)$. The expression

$$\Gamma(G) + \Gamma(H) = (\Sigma\gamma(u_i)) + (\Sigma\gamma(t_i))$$

contains both $\gamma(u_j)$ and $\gamma(t_n)$, so using $\gamma(u_j) + \gamma(t_n) = \gamma(t_n)$ we can delete the occurrence of $\gamma(u_j)$ without affecting the value of $\Gamma(G) + \Gamma(H)$. In this way we can delete all elements $\gamma(w)$ where $w \in (\{u_i | i < k\} \cup \{t_i | i < l\}) - \{v_i | i < m\}$ which occur in the sum for $\Gamma(G) + \Gamma(H)$ without affecting the value of this sum. Hence we get $\Gamma(G \cup H) = \Gamma(G) + \Gamma(H)$.

Now we show that Γ preserves multiplication. We defined multiplication so that $G \cdot H = \{\bar{x} + \bar{y} \,|\, \bar{x} \in G,\ \bar{y} \in H\}$. The minimal elements of this filter are precisely the elements of the form $\bar{x} + \bar{y}$ where $\bar{x}$ is minimal in G and $\bar{y}$ is minimal in H. If M denotes the set of minimal members of G and N denotes the set of minimal members of H, then

$$\Gamma(G \cdot H) = \sum_{\langle \bar{x}, \bar{y} \rangle \in M \times N} \gamma(\bar{x} + \bar{y}).$$

Now

$$\begin{aligned} \gamma(\bar{x} + \bar{y}) &= g_0^{x_0+y_0} \cdots g_{n-1}^{x_{n-1}+y_{n-1}} \\ &= (g_0^{x_0} \cdots g_{n-1}^{x_{n-1}})(g_0^{y_0} \cdots g_{n-1}^{y_{n-1}}) \\ &= \gamma(\bar{x})\gamma(\bar{y}). \end{aligned}$$

Hence

$$\begin{aligned} \Gamma(G \cdot H) &= \textstyle\sum_{\langle \bar{x}, \bar{y} \rangle \in M \times N} \gamma(\bar{x} + \bar{y}) \\ &= \textstyle\sum_{\langle \bar{x}, \bar{y} \rangle \in M \times N} \gamma(\bar{x})\gamma(\bar{y}) \\ &= (\textstyle\sum_M \gamma(\bar{x}))(\textstyle\sum_N \gamma(\bar{y})) \\ &= \Gamma(G)\Gamma(H). \end{aligned}$$

To see that Γ is surjective, choose $p \in F_n$. Using the semiring laws we can write p as a sum of monomials. Fix one such expression $p = \Sigma_{i<r} m_i$ which involves a minimum number of monomials. Each m_i is clearly in the image of γ since $g_0^{e_0} \cdots g_{n-1}^{e_{n-1}} = \gamma(t)$ for $t = (e_0, \ldots, e_{n-1})$. So we may define $t_i \in \omega^n$ to be an element such that $\gamma(t_i) = m_i$. Observe that if $t_j \leq t_k$ for some $j \neq k$, then the expression $p = \Sigma_{i<r} m_i$ does not contain a minimum number of monomials since

$$t_j \leq t_k \to \gamma(t_j) + \gamma(t_k) = \gamma(t_j)$$

or $m_j + m_k = m_j$ and one could shorten $\Sigma_{i<r} m_i$ without changing it by simply deleting m_k. Hence the tuples $\{t_i \mid i < r\}$ that we have defined are pairwise incomparable. Let G be the order-filter generated by $\langle t_0, \ldots, t_{r-1} \rangle$. Then

$$\Gamma(G) = \Sigma_{i<r}\gamma(t_i) = \Sigma_{i<r} m_i = p.$$

Since p was chosen arbitrarily, Γ is onto. Thus, $\langle \mathcal{F}(\omega^n); \cdot, +, \omega^n, \emptyset \rangle$ is indeed free on $\{F_i \mid i < n\}$. The theorem follows from this. $\square$

From Theorem 4.7 we find that dual bqos are not very "wide" (they have no infinite antichains) and not very "tall" (they have no infinite ascending chains). But they might be "deep" as the dual of any large ordinal witnesses. We shall find out in the next theorem that the underlying semilattice of a finitely generated, commutative semiring satisfying $1 + r = 1$ is a dual bqo, but it is one which is not very "deep".

THEOREM 4.9 *If* $\mathbf{R}$ *is an* n*-generated, commutative semiring satisfying* $1 + r = 1$*, then the following statements are true about the semilattice order on* R*.*

(i) *The underlying semilattice of* $\mathbf{R}$ *is a homomorphic image of* $\langle \mathcal{F}(\omega^n); \cup \rangle$*. In particular, the semilattice order satisfies the ascending chain condition and contains no infinite antichains.*

(ii) *The dual order contains no strictly increasing chain of order-type* $> \omega^n + 1$*.*

Proof: In this proof we use ω^n to refer to the ordinal ω^n and also to refer to the poset which is a product of n copies of ω and has the product ordering. The context will indicate which meaning we intend.

Only the last statement of the theorem needs to be proved. For this it suffices to show that the dual order on $\mathcal{F}(\omega^n)$ has no strictly increasing chain of order-type $> \omega^n + 1$. If the claim is false, then the dual order on $\mathcal{F}(\omega^n)$ has a strictly increasing chain of order-type $\omega^n + 2$. Suppose that

$$F_0 \supset F_1 \supset \cdots \supset F_{\omega^n} \supset F_{\omega^n+1}$$

is such a chain. Then we may choose $t_\alpha \in F_\alpha - F_{\alpha+1}$ and in this way obtain a sequence of length $\omega^n + 1$ consisting of n-tuples $t_0, \ldots, t_{\omega^n} \in \omega^n$ with the property that $\alpha < \beta \to t_\alpha \not\geq t_\beta$. We shall argue that ω^n contains no such sequence.

Our argument will be by induction on n with the case $n = 0$ being trivial. (Since this leads to a sequence t_0, t_1 with $t_0 \not\geq t_1$ in a 1-element ordered set.) Our inductive hypothesis shall be: ω^k contains no sequence $t_0, \ldots, t_{\omega^k}$ with the property that $\alpha < \beta \to t_\alpha \not\geq t_\beta$. Suppose that $n > 0$ is the first value of k where our hypothesis fails and assume that $t_{\omega^n} = (m_0, \cdots, m_{n-1})$. Since $t_\alpha \not\geq t_{\omega^n}$ when $\alpha < \omega^n$ we get that for each t_α, $\alpha < \omega^n$, there is at least one coordinate $i < n$ such that $(t_\alpha)_i < m_i$. Let $S_{i,h}$ denote the subsequence of indices α such that $(t_\alpha)_i = h < m_i$. Every $\alpha < \omega^n$ belongs to some subsequence $S_{i,h}$ and the number of subsequences is finite (since $i < n$ and $h < m_i$). But whenever the ordinal ω^n is expressed as a union of finitely many ordered sets, at least one is a chain of order-type ω^n. Choose i and h so that $S_{i,h}$ has order-type ω^n. Each t_α with $\alpha \in S_{i,h}$ has $(t_\alpha)_i = h$. By deleting the i^{th} coordinate in each t_α, $\alpha \in S_{i,h}$, we obtain a sequence of elements $u_0, u_1, \ldots$ in ω^{n-1} of length ω^n which has the property that if $\alpha < \beta$ then $u_\alpha \not\geq u_\beta$. By deleting the last part of this chain we obtain a sequence $u_0, u_1, \ldots, u_{\omega^{n-1}}$ of length $\omega^{n-1} + 1$ which has the property that if $\alpha < \beta$ then $u_\alpha \not\geq u_\beta$. This contradicts our induction hypothesis, so we are done. □

4.2 The Semiring of a Variety

It follows easily from Theorem 3.1 that if $\mathcal{V}$ is a variety of semilattice modes, then every subdirectly irreducible member of $\mathcal{V}$ is polynomially equivalent to a semimodule over some commutative semiring satisfying $1 + r = 1$. This leads one to expect or at least hope that there is a single semiring associated with $\mathcal{V}$ that acts naturally on all the subdirectly irreducible members of $\mathcal{V}$. We shall find in this subsection that this is so: we shall define the semiring of "coefficients of terms".

In the following definition R is the subuniverse of $\mathbf{F}_\mathcal{V}(x, y)$ consisting of all $t \in F_\mathcal{V}(x, y)$ such that $t + y = t$ (or $t \geq y$). If $t \in F_\mathcal{V}(x, y)$, then we write e_t for the endomorphism of $\mathbf{F}_{\mathcal{V}}(x, y)$ determined by $x \mapsto t$, $y \mapsto y$. For $s, t \in F_{\mathcal{V}}(x, y)$ we write $s \circ t$ to denote $e_t(s)$.

Definition 4.10 If $\mathcal{V}$ is a variety of semilattice modes, let $\mathbf{R}(\mathcal{V})$ be the algebra of type $\langle 2, 2, 0, 0\rangle$ given by $\langle R; \circ, +, x + y, y\rangle$.

THEOREM 4.11 *If $\mathcal{V}$ is a variety of semilattice modes, then $\mathbf{R}(\mathcal{V})$ is a commutative semiring satisfying $1 + r = 1$.*

Proof: R contains $x + y$ since $(x + y) + y = x + y$. R contains y since $y + y = y$. R is closed under $+$ since R is a subuniverse of $\mathbf{F}_{\mathcal{V}}(x, y)$ and $+$ is a term operation of $\mathbf{F}_{\mathcal{V}}(x, y)$. To see that R is closed under $\circ$, note that for $s, t \in R$ we have

$$(s \circ t) + y = e_t(s) + e_t(y) = e_t(s + y) = e_t(s) = s \circ t.$$

Hence $\mathbf{R}(\mathcal{V})$ is closed under its operations.

We begin verifying that $\mathbf{R}(\mathcal{V})$ satisfies the required equations by first examining those that do not involve $\circ$. Since $+$ is a semilattice operation, it is associative and commutative. The fact that $t + y = t$ for all $t \in R$ is just the unit law for $+$. To see that $1 + r = 1$ for all $r \in R$, choose a binary term $r(x, y)$ representing r. Then $1 + r$ is represented by $(x + y) + r(x, y)$ and

$$\begin{aligned}(x + y) + r(x, y) &= r(x + y, x + y) + r(x, y)\\ &= r((x + y) + x, (x + y) + y)\\ &= r(x + y, x + y)\\ &= x + y.\end{aligned}$$

Now we check the equations that involve $\circ$. Let $r(x, y)$, $s(x, y)$ and $t(x, y)$ be binary terms representing $r, s, t \in R$. Then $r \circ (s \circ t)$ and $(r \circ s) \circ t$ are both represented by $r(s(t(x, y), y), y)$. Hence they are equal elements of R. Since $r \circ s$ is represented by $r(s(x, y), y) = r(s(x, y), s(y, y)) = s(r(x, y), r(y, y)) = s(r(x, y), y)$ which represents $s \circ r$, we get that $\circ$ is commutative. $1 \circ r$ is represented by $(x + y) \circ r(x, y) = r(x, y) + y$ $= r(x, y)$, so the unit law for $\circ$ holds. $r \circ 0$ is represented by $r(x, y) \circ y = r(y, y) = y$, so the absorptive law for 0 holds. Finally, $r \circ (s + t)$ is represented by

$$r(s(x, y) + t(x, y), y) = r(s(x, y) + t(x, y), y + y) = r(s(x, y), y) + r(t(x, y), y)$$

which represents $r \circ s + r \circ t$. The other distributive law follows from this one and commutativity. □

If one refers back to Lemma 3.5 one finds that the semiring $\mathbf{R}(\mathcal{V})$ may be identified with the semiring of "coefficients of terms up to $\mathcal{V}$-equivalence" by identifying each equivalence class of coefficients with the element of $\mathbf{F}_{\mathcal{V}}(x, y)$ that it represents. In that lemma and the paragraph preceding it we defined 0, 1, $+$ and product (written $\hat{s}\hat{t}$ there). We reiterate that we shall treat our previously defined notion of a coefficient loosely by considering coefficients to be elements of $\mathbf{R}(\mathcal{V})$.

THEOREM 4.12 *If $\mathcal{V}$ is a variety of semilattice modes and $\mathbf{A} \in \mathcal{V}$ is subdirectly irreducible with least element 0, then $\langle A; +, 0, \lambda_r (r \in R)\rangle$ is an $\mathbf{R}(\mathcal{V})$-semimodule polynomially equivalent to $\mathbf{A}$. ($\lambda_r(a) \stackrel{\text{def}}{=} r^{\mathbf{A}}(a,0)$ where $r(x,y)$ is any term representing r.)*

Proof: This result is essentially Theorem 3.1 (v). The only difference here is that we have introduced the terminology of semimodules. □

THEOREM 4.13 *If $\mathbf{A}$ is a subdirectly irreducible semilattice mode with least element 0, then $\mathbf{R}(\mathsf{V}(\mathbf{A})) \cong \langle U; \circ, +, id_A, 0\rangle$ where $U = \{p(x) \in \mathrm{Pol}_1\mathbf{A} \mid p(0) = 0\}$.*

Proof: The map $\mathbf{R}(\mathsf{V}(\mathbf{A})) \to \langle U; \circ, +, id_A, 0\rangle : r \mapsto r^{\mathbf{A}}(x,0)$ is the required isomorphism. What this expression means is that for $r \in R(\mathsf{V}(\mathbf{A}))$ we choose a binary term $r(x,y)$ so that $r = r^{\mathbf{F}}(x,y)$ in $\mathbf{F} = \mathbf{F}_{\mathsf{V}(\mathbf{A})}(x,y)$. Then we map r to the polynomial $r^{\mathbf{A}}(x,0)$.

The ring $\mathbf{R}(\mathsf{V}(\mathbf{A}))$ was defined so as to make this map a homomorphism. The map is onto by Theorem 3.1 (iv). To see that it is 1-1, assume that $r, s \in R(\mathsf{V}(\mathbf{A}))$ and that $r(x,y)$ and $s(x,y)$ are binary terms representing these elements. Assume that $r^{\mathbf{A}}(x,0) = s^{\mathbf{A}}(x,0)$. Then

$$r^{\mathbf{A}}(x,y) = r^{\mathbf{A}}(x,0) + y = s^{\mathbf{A}}(x,0) + y = s^{\mathbf{A}}(x,y).$$

Thus $r = s$ in $R(\mathsf{V}(\mathbf{A}))$ and the given map is an isomorphism. □

The next lemma concerns varieties of semilattice modes with finitely many basic operations. If $\mathcal{V}$ is such a variety and $\mathcal{V}$ has basic operations $f^0, \ldots, f^{k-1}$ with arities $n_0, \ldots, n_{k-1}$, then we call $\Sigma_{i<k} n_i$ the **gross arity** of $\mathcal{V}$. We call $\Sigma_{i<k}(n_i - 1)$ the **net arity** of $\mathcal{V}$. (These terms were suggested by a referee.)

LEMMA 4.14 *Let $\mathcal{V}$ be a variety of semilattice modes of finite type. If G is the gross arity of $\mathcal{V}$, then $\mathbf{R}(\mathcal{V})$ may be generated by $\leq G$ elements. If $\mathcal{V}$ is generated by a subdirectly irreducible mode and N is the net arity of $\mathcal{V}$, then $\mathbf{R}(\mathcal{V})$ may be generated by $\leq N$ elements.*

Proof: First, we argue that $\mathbf{R}(\mathcal{V})$ is generated by the coefficients of the basic operations.

For any binary term $r(x,y)$ the term $r(x,y) + y$ represents an element of $\mathbf{R}(\mathcal{V})$; in fact, it represents the first coefficient of $r(x,y)$. Conversely, if $r \in R(\mathcal{V})$ and $r(x,y)$ is a term which represents r, then $r(x,y) + y$ also represents r. Hence, the elements of $R(\mathcal{V})$ are precisely the elements of $F_{\mathcal{V}}(2)$ which are represented by terms of the form $r(x,y) + y$. We will argue by induction on the complexity of $r(x,y)$ that the element of $R(\mathcal{V})$ represented by $r(x,y) + y$ is contained in the subsemiring of $\mathbf{R}(\mathcal{V})$ generated by the coefficients of the basic operations.

If $r(x,y) = x$ or y, then $r(x,y) + y = x + y$ or $r(x,y) + y = y$ and these represent the elements $1, 0 \in R(\mathcal{V})$, respectively. These elements of $R(\mathcal{V})$ are generated for free, so there is nothing to check here. Next, suppose that

$$r(x,y) = f(s_0(x,y), \ldots, s_{k-1}(x,y))$$

where f is a basic operation and, for each i, $s_i(x,y)+y$ represents an element of $R(\mathcal{V})$ contained in the subsemiring generated by the coefficients of the basic operations. Then, recalling that $\hat{f}_i(x,y) = f(y,\ldots,y,x,y,\ldots,y)+y$, we have

$$\begin{aligned} r(x,y)+y &= f(s_0(x,y),\ldots,s_{k-1}(x,y))+y \\ &= (f(s_0(x,y),\ldots,s_{k-1}(x,y))+f(y,\ldots,y))+y \\ &= f(s_0(x,y)+y,\ldots,s_{k-1}(x,y)+y)+y \\ &= \Sigma_{i<k}\hat{f}_i(s_i(x,y)+y,y). \end{aligned}$$

Each $\hat{f}_i(x,y)$ represents a coefficient of a basic operation and, by induction, each $s_i(x,y)+y$ represents an element in the subsemiring of $\mathbf{R}(\mathcal{V})$ generated by the coefficients of basic operations. Hence, $r(x,y)+y$ also represents an element in this subsemiring. We conclude that $\mathbf{R}(\mathcal{V})$ is generated by the coefficients of the basic operations. There are $\le G$ coefficients of basic operations which are distinct, so the first statement of the lemma is proved.

If $\mathcal{V}$ is generated by a subdirectly irreducible, then by the argument in Theorem 3.3 each term has at least one coefficient equal to 1. Now $\mathbf{R}(\mathcal{V})$ is generated by the coefficients of the basic operations which are different than 1 since 1 is generated for free. There are $\le N$ of these coefficients, so we are done. □

COROLLARY 4.15 *If $\mathcal{V}$ is a variety of semilattice modes of finite type, then $\mathcal{V}$ is residually countable.*

Proof: If $\mathcal{V}$ is not residually countable, then $\mathcal{V}$ has a subvariety generated by an uncountable subdirectly irreducible. Thus it suffices to show that if $\mathcal{V} = \mathsf{V}(\mathbf{A})$ where $\mathbf{A}$ is subdirectly irreducible, then $\mathbf{A}$ is countable.

By Lemma 4.14, the semiring of $\mathcal{V}$ is finitely generated. By Corollary 4.6, the variety of $\mathbf{R}(\mathcal{V})$-semimodules is residually countable. But Theorem 4.12 proves that $\mathbf{A}$ is polynomially equivalent to a subdirectly irreducible $\mathbf{R}(\mathcal{V})$-semimodule. Hence $\mathbf{A}$ is countable. □

4.3 Further Applications

In this section we show how some interesting properties of a variety of semilattice modes are connected to the structure of $\mathbf{R}(\mathcal{V})$. First we explain how to construct a canonical cogenerating algebra for $\mathcal{V}$ from $\mathbf{R}(\mathcal{V})$. By a **cogenerating algebra**, we mean an algebra $\mathbf{C}$ such that $\mathcal{V} = \mathsf{SP}(\mathbf{C})$. Thus a cogenerating algebra for $\mathcal{V}$ is an algebra in $\mathcal{V}$ in which it is possible to embed every subdirectly irreducible of $\mathcal{V}$. The underlying semilattice of the cogenerating algebra will be the collection of annihilator ideals of $\mathbf{R}(\mathcal{V})$ under intersection. That this semilattice underlies a cogenerating algebra for $\mathcal{V}$ severely restricts the structure of the subdirectly irreducible members of $\mathcal{V}$ and yields alternate proofs of Corollary 4.15 and the first claim of Theorem 3.8. Next we prove that **Con** $\mathbf{R}(\mathcal{V})$ is isomorphic to the lattice of equational theories extending the theory of $\mathcal{V}$ (or dually isomorphic to the lattice of subvarieties of $\mathcal{V}$.) Finally we show a connection between $\mathbf{R}(\mathcal{V})$ and the free spectrum of $\mathcal{V}$.

If $\mathcal{V}$ is a variety of semilattice modes, let $\mathcal{I}$ be the set of annihilator ideals of $\mathbf{R}(\mathcal{V})$ and define $a \oplus b = a \cap b$ for $a, b \in \mathcal{I}$. For each $r \in \mathbf{R}(\mathcal{V})$ and each $a \in \mathcal{I}$ let $r^{-1}(a) =$

$\{x \in R(\mathcal{V}) | r \circ x \in a\}$. For $a \in \mathcal{I}$ it is the case that $r^{-1}(a) \in \mathcal{I}$ and $a \subseteq r^{-1}(a)$. Further, since $\mathbf{R}(\mathcal{V})$ is commutative, each r^{-1} commutes with each s^{-1} when $r, s \in \mathbf{R}(\mathcal{V})$. Indeed, considering $\mathbf{S} = \langle \mathcal{I}; \oplus \rangle$ as a join semilattice, $U = \{r^{-1}(x) | r \in R(\mathcal{V})\}$ is a collection of decreasing, commuting endomorphisms of $\mathbf{S}$.

If f is an n-ary basic operation symbol for $\mathcal{V}$, then we define a corresponding n-ary operation $[f]$ on $\mathcal{I}$ by

$$[f](I_0, \ldots, I_{n-1}) = \hat{f}_0^{-1}(I_0) \oplus \cdots \oplus \hat{f}_{n-1}^{-1}(I_{n-1})$$

for $I_j \in \mathcal{I}$, $j < n$. Here we use the notation introduced prior to Lemma 3.5; writing $\hat{f}_i$ to denote the i^{th} coefficient of f. We may equip $\mathcal{I}$ with operations $[f]$, f a basic operation symbol for $\mathcal{V}$, and obtain an algebra of the same type as $\mathcal{V}$. Let us write $\mathcal{I}(\mathcal{V})$ to denote this algebra.

THEOREM 4.16 *$\mathcal{I}(\mathcal{V}) \in \mathcal{V}$.*

Proof: We extend our square bracket notation to terms as follows: if t is a k-ary term of $\mathcal{V}$, then we define

$$[t](I_0, \ldots, I_{k-1}) = \hat{t}_0^{-1}(I_0) \oplus \cdots \oplus \hat{t}_{k-1}^{-1}(I_{k-1}).$$

Observe that the assignment $t \mapsto [t]$ preserves composition. For if s is an n-ary $\mathcal{V}$-term and t^i is k-ary, then by Lemma 3.5 we get that the i^{th} coefficient of $s(t^0, \ldots, t^{(n-1)})(\bar{x})$ is just

$$\hat{s}_0 \circ \hat{t}_i^0 + \cdots + \hat{s}_{n-1} \circ \hat{t}_i^{(n-1)}.$$

But

$$\begin{aligned}
(\Sigma_{j<n} \hat{s}_j \circ \hat{t}_i^j) \circ r \in I &\leftrightarrow (\Sigma_{j<n} \hat{s}_j \circ \hat{t}_i^j \circ r) \in I \\
&\leftrightarrow (\forall j) \hat{s}_j \circ \hat{t}_i^j \circ r \in I \\
&\leftrightarrow (\forall j) r \in (\hat{s}_j \circ \hat{t}_i^j)^{-1}(I) \\
&\leftrightarrow r \in \textstyle\bigcap_{j<n} \hat{s}_j^{-1}((\hat{t}_i^j)^{-1}(I)) \\
&\leftrightarrow r \in (\hat{s}_0^{-1}(\hat{t}_i^0)^{-1} \oplus \cdots \oplus \hat{s}_{n-1}^{-1}(\hat{t}_i^{(n-1)})^{-1})(I).
\end{aligned}$$

Hence

$$\widehat{s(\bar{t})}_i^{-1} = \hat{s}_0^{-1}(\hat{t}_i^0)^{-1} \oplus \cdots \oplus \hat{s}_{n-1}^{-1}(\hat{t}_i^{(n-1)})^{-1}.$$

From this and Lemma 3.5 we get that

$$[s(\bar{t})] = [s](\overline{[t]}).$$

This verifies that the assignment $t \mapsto [t]$ is a homomorphism from the clone of $\mathcal{V}$ to the clone of $\mathcal{I}(\mathcal{V})$ which maps the basic operations of $\mathcal{V}$ to the basic operations of $\mathcal{I}(\mathcal{V})$. Therefore, $\mathcal{I}(\mathcal{V}) \in \mathcal{V}$. $\square$

THEOREM 4.17 *If $\mathcal{V}$ is a variety of semilattice modes and $\mathbf{A} \in \mathcal{V}$ is subdirectly irreducible, then $\mathbf{A}$ is embeddable in $\mathcal{I}(\mathcal{V})$.*

Proof: Since $\mathbf{A}$ is subdirectly irreducible, there is an element $0 \in A$ which is the least element in the semilattice ordering of $\mathbf{A}$ and there is an element $u \in A$ which is the second least element. For each $a \in A$ let $I_a = \{r \in R(\mathcal{V}) | r(a) = 0\}$. Clearly each I_a is an annihilator ideal. Furthermore, $I_0 = R(\mathcal{V})$ and $I_u \subseteq R(\mathcal{V}) - \{1\}$. If $\phi : \mathbf{A} \to \mathcal{I}(\mathcal{V})$ is the function $a \mapsto I_a$, then $\phi(0) \neq \phi(u)$. Hence, if we show that ϕ is a homomorphism, then it will follow that ϕ is 1-1, since the monolith of $\mathbf{A}$ is $\mathrm{Cg}(0, u)$.

To show that ϕ is a homomorphism, we must show that for any basic operation f it is the case that $\phi(f(a_0, \ldots, a_{n-1})) = [f](\phi(a_0), \ldots, \phi(a_{n-1}))$ or

$$I_{f(\bar{a})} = \hat{f}_0^{-1}(I_{a_0}) \oplus \cdots \oplus \hat{f}_{n-1}^{-1}(I_{a_{n-1}}).$$

We verify this with the computation

$$\begin{aligned} r \in I_{f(\bar{a})} &\leftrightarrow r(f(\bar{a})) = 0 \\ &\leftrightarrow r^{\mathbf{A}}(f(\bar{a}), 0) = 0 \\ &\leftrightarrow r^{\mathbf{A}}(\hat{f}_0(a_0, 0) + \cdots + \hat{f}_{n-1}(a_{n-1}, 0), 0) = 0 \\ &\leftrightarrow r^{\mathbf{A}}(\hat{f}_0(a_0, 0), 0) + \cdots + r^{\mathbf{A}}(\hat{f}_{n-1}(a_{n-1}, 0), 0) = 0 \\ &\leftrightarrow (\forall i) r^{\mathbf{A}}(\hat{f}_i(a_i, 0), 0) = 0 \\ &\leftrightarrow (\forall i)(\hat{f}_i r)(a_i, 0) = 0 \\ &\leftrightarrow (\forall i) r \in \hat{f}_i^{-1}(I_{a_i}) \\ &\leftrightarrow r \in \textstyle\bigcap_{i<n} \hat{f}_i^{-1}(I_{a_i}) \\ &\leftrightarrow r \in \hat{f}_0^{-1}(I_{a_0}) \oplus \cdots \oplus \hat{f}_{n-1}^{-1}(I_{a_{n-1}}). \end{aligned}$$

We use the fact that for $x \in A$ we have defined $r(x)$ to mean $r^{\mathbf{A}}(x, 0)$. This finishes the proof. $\square$

Theorems 4.16 and 4.17 together imply that

$$\mathbf{HSP}(\mathcal{I}(\mathcal{V})) \subseteq \mathcal{V} \subseteq \mathbf{SP}(\mathcal{I}(\mathcal{V})).$$

It follows that both inclusions are equalities. We call $\mathcal{I}(\mathcal{V})$ the **canonical cogenerator** for $\mathcal{V}$. The canonical cogenerator for $\mathcal{V}$ is quite useful for studying the subdirectly irreducible algebras in $\mathcal{V}$. We work out an exercise which shows this.

Exercise. *Find all subdirectly irreducible groupoid modes which have a compatible semilattice operation.* (We say that $\mathbf{A}$ has a compatible semilattice operation if there is a homomorphism $+ : \mathbf{A}^2 \to \mathbf{A}$ which satisfies $x + x = x$, $x + y = y + x$ and $x + (y + z) = (x + y) + z$.) If a groupoid mode has a compatible semilattice operation, then adjoining this operation creates a semilattice mode. A subdirectly irreducible mode expanded in this way remains subdirectly irreducible. If we find all subdirectly irreducible members of the variety $\mathcal{V}$ of modes with two binary operations, $x + y$ and $t(x, y)$, subject only to equations which force $\mathcal{V}$ to be a variety of semilattice modes with semilattice operation $x + y$, then we will almost be done. If $\{\mathbf{A}_i | i \in I\}$ is the set of subdirectly irreducible algebras obtained, then to solve the exercise we need to form the reducts of the $\mathbf{A}_i$s to the operation $t(x, y)$ and keep only those reducts which remain subdirectly irreducible.

The first step is to compute $\mathbf{R}(\mathcal{V})$. This semiring is generated by the coefficients of the basic operations $x + y$ and $t(x, y)$. Since $x + y$ is the semilattice operation, both its first and second coefficients are 1. If $t(x, y)$ is represented as $a \bullet x + b \bullet y$, then $a+b=1$ since $t(x,x)=x$. It follows that $\mathbf{R}(\mathcal{V})$ is a homomorphic image of the semiring presented by $\langle a,b|a+b=1\rangle$. Let $\mathbf{S}$ denote the latter semiring. We claim that

$$a^m b^n + a^r b^s = a^{\min\{m,r\}} b^{\min\{n,s\}}$$

is a consequence of $a+b=1$. To prove this, let $u=\min\{m,r\}$ and let $v=\min\{n,s\}$. Depending on the order relationships between m and r or n and s, we can write

$$a^m b^n + a^r b^s = \begin{cases} a^u b^v (1 + a^{r-m} b^{s-n}), \\ a^u b^v (a^{m-r} b^{n-s} + 1), \\ a^u b^v (a^{m-r} + b^{s-n}), \text{or} \\ a^u b^v (b^{n-s} + a^{r-m}), \end{cases}$$

where all exponents are nonnegative. Since $\mathbf{S} \models 1+r=1$, we already have the desired result in the first two cases. To prove it in the other cases, it will suffice to show that $a+b=1$ entails $a^i+b^j=1$ for all $i,j\geq 0$. Since $a+b=1$ and $\mathbf{S}\models 1+r=1$, we have

$$\begin{aligned} 1 &= a+1 \\ &= a+(a+b)^j \\ &= a+(a^j+a^{j-1}b+\cdots+ab^{j-1}+b^j) \\ &= a(1+a^{j-1}+a^{j-2}b+\cdots+b^{j-1})+b^j \\ &= a+b^j. \end{aligned}$$

So, $a+b=1$ entails $a+b^j=1$ for any $j\geq 0$. Applying the same argument to $a+b^j=1$ with a and b^j in place of b and a we obtain that $a^i+b^j=1$ for any $i,j\geq 0$. This can be used above to simplify both $a^u b^v(a^{m-r}+b^{s-n})$ and $a^u b^v(b^{n-s}+a^{r-m})$ to $a^u b^v$.

Now we have that every element of $\mathbf{S}$ may be expressed as 0 or $a^m b^n$, $m,n\geq 0$. From this it is not too hard to show that, in any proper homomorphic image of $\mathbf{S}$, either a or b has finite order. If $\mathbf{R}(\mathcal{V})$ were a proper homomorphic image of $\mathbf{S}$, then by symmetry both a and b would be of finite order. This would force $\mathbf{R}(\mathcal{V})$ to be finite and $\mathcal{V}$ to be locally finite. But in fact this is not the case, since for $N=1$ the algebra described in Example 6 below generates a non-locally finite subvariety of $\mathcal{V}$. Hence $\mathbf{S}$ $=\mathbf{R}(\mathcal{V})$.

Next, we need to calculate $\mathcal{I}(\mathcal{V})$. Since $\mathcal{V}$ is of finite type, $\mathbf{R}(\mathcal{V})$ is Noetherian and so every annihilator ideal is finitely generated. We will use the symbol (m,n) to denote the ideal $\{r\in R(\mathcal{V})|r\leq a^m b^n\}$ and ∞ to denote the ideal $\{0\}$. The join-semilattice order given by $\oplus := \bigcap$ is the reverse of the inclusion order, so

$$\begin{aligned} (m,n)\leq(r,s) &\leftrightarrow a^r b^s \leq a^m b^n \\ &\leftrightarrow m=\min\{r,m\} \;\&\; n=\min\{s,n\} \\ &\leftrightarrow m\leq r \;\&\; n\leq s. \end{aligned}$$

The ordering used in the last line is the usual order on the natural numbers. Also $(m,n)\leq\infty$ for all m,n. Thus, the join-semilattice order on $\mathcal{I}(\mathcal{V})$ agrees with the usual product order on $\omega\times\omega$ with ∞ placed above all other elements.

Now $[t]=a^{-1}(x)\oplus b^{-1}(y)$, hence

$$\begin{aligned} [t]((m,n),(r,s)) &= a^{-1}(m,n)\oplus b^{-1}(r,s) \\ &= (m-1,n)+(r,s-1) \\ &= (\min\{m-1,r\},\min\{n,s-1\}). \end{aligned}$$

In particular, this shows that $[t]((m,n),(0,0)) = (m-1,n)$ and $[t]((0,0),(m,n)) = (m,n-1)$. From this, one can show that the subalgebra generated by (m,n) and $(0,0)$ contains every element in the interval $[(0,0),(m,n)]$. We also have $a^{-1}(\infty) = \infty = b^{-1}(\infty)$. This information completely determines $[t]$. Since $[+] = \oplus$, we have a complete description of $\mathcal{I}(\mathcal{V}) = \langle \mathcal{I}; \oplus, [t] \rangle$.

If $\mathbf{A} \in \mathcal{V}$ is subdirectly irreducible, then Theorem 4.17 describes an embedding ϕ of $\mathbf{A}$ into $\mathcal{I}(\mathcal{V})$. For 0 equal to the least element of $\mathbf{A}$ we have $\phi(0) = I_0 = R = (0,0)$. For u equal to the second least element of $\mathbf{A}$ we have that $\phi(u) > \phi(0)$ and that $\phi(\{0,u\})$ is a subuniverse of $\mathcal{I}(\mathcal{V})$ (since $\{0,u\}$ is a subuniverse of $\mathbf{A}$). There are two possible cases. Case (i) is where $\phi(u) = \infty$. In this case, if $v \in A - \{0,u\}$, then $v > u$ so $\phi(v) > \phi(u) = \infty$ which is impossible. Thus $A = \{0,u\}$ and $\mathbf{A} \cong \phi(\mathbf{A})$. The algebra $\phi(\mathbf{A})$ in this case is the subalgebra of $\mathcal{I}(\mathcal{V})$ generated by $\{(0,0),\infty\}$ and a short calculation establishes that both $\oplus$ and $[t]$ interpret as the same operation: the join-semilattice operation with respect to the order $(0,0) < \infty$. Now assume that we are not in Case (i). Case (ii) is where $\phi(u) \neq \infty$, so $\phi(u) = (m,n)$ for some m and n. Now $\{\phi(0),\phi(u)\} = \{(0,0),(m,n)\}$ is a 2-element subuniverse of $\mathcal{I}(\mathcal{V})$. But, as observed in the last paragraph, this subuniverse contains the entire interval $[(0,0),(m,n)]$. Hence $(m,n) = (0,1)$ or $(m,n) = (1,0)$. Both cases can be handled symmetrically, so assume that $\phi(u) = (0,1)$. If there exists $v \in A - \{0,u\}$, then we have $v > u$ and so $\phi(v) > \phi(u) = (0,1)$. Assume that $\phi(v) = (r,s)$. The entire interval $[(0,0),(r,s)]$ is in the subuniverse $\phi(A)$ of $\mathcal{I}(\mathcal{V})$, so if $r \neq 0$, then we arrive at the contradiction $(1,0) \in \phi(A)$. (This is a contradiction for the following reason. Every element of A is comparable with u, so every element of $\phi(A)$ is comparable with $\phi(u) = (0,1)$.) We conclude that if $\phi(v) = (r,s)$, then $r = 0$. It follows that

$$\phi(A) \subseteq (\{0\} \times \omega) \cup \{\infty\}$$

and further, that $\phi(A) - \{\infty\}$ equals $\{0\} \times I$ where I is an initial segment of ω. In particular, this implies that the semilattice order on any subdirectly irreducible is that of an ordinal $\leq \omega + 1$.

It turns out that any subalgebra of $\mathcal{I}(\mathcal{V})$ whose universe is of the form $\{0\} \times I$ or $(\{0\} \times I) \cup \{\infty\}$, where I is an initial segment of ω, is subdirectly irreducible in $\mathcal{V}$. When $I = \{0\}$, this describes the algebra in Case (i) of the last paragraph and when $I \neq \{0\}$ this describes the algebras in Case (ii). Hence we have a complete list of the subdirectly irreducible members of $\mathcal{V}$. They are precisely those subalgebras of $\mathcal{I}(\mathcal{V})$ whose universe is of the form

- (i) $\{0\} \times I$,
- (ii) $(\{0\} \times I) \cup \{\infty\}$,
- (iii) $I \times \{0\}$ or
- (iv) $(I \times \{0\}) \cup \{\infty\}$

where I is an initial segment of ω. At this point it is worth looking at the table for the operation $[t] = a^{-1}(x) \oplus b^{-1}(y)$ of $\mathcal{I}(\mathcal{V})$ restricted to the subalgebra $(\{0\} \times \omega) \cup \{\infty\}$.

In this table we write $\mathbf{n}$ in place of $(0, n)$ to simplify the table.

$[t]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$\cdots$	∞
0	**0**	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$	∞
1	**1**	**1**	**1**	**2**	**3**	**4**	**5**	$\cdots$	∞
2	**2**	**2**	**2**	**2**	**3**	**4**	**5**	$\cdots$	∞
3	**3**	**3**	**3**	**3**	**3**	**4**	**5**	$\cdots$	∞
4	**4**	**4**	**4**	**4**	**4**	**4**	**5**	$\cdots$	∞
5	**5**	**5**	**5**	**5**	**5**	**5**	**5**	$\cdots$	∞
6	**6**	**6**	**6**	**6**	**6**	**6**	**6**	$\cdots$	∞
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
∞	∞	∞	∞	∞	∞	∞	∞	$\cdots$	∞

Our conclusions up to this point imply that any subdirectly irreducible whose universe is $\{0\} \times \alpha$, where $\alpha \leq \omega$ is an ordinal, has the same (join-)semilattice order as the ordinal α and has $[t]$-table equal to the $\alpha \times \alpha$ upper left corner in the above table. If a subdirectly irreducible has universe $(\{0\} \times \alpha) \cup \{\infty\}$, then the semilattice order on α is that of the ordinal α with the further stipulation that $i < \infty$ for all $i < \alpha$. The $[t]$-table is the restriction of the above table to $(\{0\} \times \alpha) \cup \{\infty\}$. Corresponding statements involving the dual table hold when the universe is of the form $\alpha \times \{0\}$ or $(\alpha \times \{0\}) \cup \{\infty\}$.

To finish the exercise, we must take the reducts of the subdirectly irreducible algebras of $\mathcal{V}$ to the t-operation only. The collection of algebras which remain subdirectly irreducible under t is a full list of (isomorphism types of) subdirectly irreducible groupoid modes with a compatible semilattice operation. We leave it to the reader to verify that *every* subdirectly irreducible in $\mathcal{V}$ has a subdirectly irreducible reduct to the operation t. Hence the groupoid operation table of any subdirectly irreducible groupoid mode with a compatible semilattice operation is contained in the previously displayed table or its dual.

Here is another immediate consequence of Theorem 4.17.

COROLLARY 4.18 *If* $\mathbf{A}$ *is a finite subdirectly irreducible semilattice mode and* $\mathcal{V} = \mathsf{V}(\mathbf{A})$, *then* $\mathbf{A} \cong \mathcal{I}(\mathcal{V})$. *Hence* $\mathcal{V} = \mathsf{SP}(\mathbf{A})$.

Proof: Theorems 3.8 and 4.13 prove that $|A| = |U| = |R(\mathcal{V})|$ in the situation described in the corollary. Since $\mathbf{R}(\mathcal{V})$ is finite, every annihilator ideal is principal and so $|R(\mathcal{V})| = |\mathcal{I}(\mathcal{V})|$. Hence $\mathbf{A}$ is finite, has the same cardinality as $\mathcal{I}(\mathcal{V})$, and $\mathbf{A}$ embeds into $\mathcal{I}(\mathcal{V})$. It follows that the embedding is an isomorphism. □

In the next result, another consequence of Theorem 4.17, $\mathcal{V}$ is a variety of semilattice modes of finite type which has net arity N.

COROLLARY 4.19 *If* $\mathbf{A} \in \mathcal{V}$ *is subdirectly irreducible, then there is a semilattice embedding of* $\langle A; + \rangle$ *into* $\langle \mathcal{F}(\omega^N); \cap \rangle$. *In particular,* $\langle A; \leq \rangle$ *is a bqo and has no strictly increasing chain of order-type* $> \omega^N + 1$.

Proof: We may assume that $\mathcal{V} = \mathsf{V}(\mathbf{A})$. The embedding established in Theorem 4.17 shows that if $\mathbf{A}$ is a subdirectly irreducible semilattice mode, then there is a semilattice embedding of $\langle A; +\rangle$ into the lattice of annihilator ideals of $\mathbf{R}(\mathcal{V})$ under intersection. By Lemma 4.14, $\mathbf{R}(\mathcal{V})$ is generated by $\leq N$ elements. Hence there is an onto homomorphism $f : \mathbf{F}_N \to \mathbf{R}(\mathcal{V})$ from the free, N-generated, commutative semiring satisfying $1 + r = 1$ onto $\mathbf{R}(\mathcal{V})$. Now f^{-1} is a 1-1 homomorphism of the semilattice of annihilator ideals of $\mathbf{R}(\mathcal{V})$ into the semilattice of annihilator ideals of $\mathbf{F}_N$. Composing this with the embedding of Theorem 4.17 gives a semilattice embedding of $\langle A; +\rangle$ into the semilattice of annihilator ideals of the free, N-generated, commutative semiring satisfying $1+r = 1$. Since $\mathbf{F}_N$ is Noetherian, each annihilator ideal is principal and we may identify each annihilator ideal with the element that generates it. Under this identification, we obtain a semilattice embedding of $\langle A; +\rangle$ into $\langle F_N; \wedge\rangle$. Here $\wedge$ is not a term of $\mathbf{F}_N$, but the greatest lower bound of any two elements exists since $\langle F_N; \leq\rangle$ is a join-semilattice which satisfies the ascending chain condition and has a least element. From Theorem 4.8 we see that we may fairly replace all instances of $\mathbf{F}_N$ in our argument with $\langle \mathcal{F}(\omega^N); \cdot, +, F, \emptyset\rangle$. In this algebra, $\wedge$ equals intersection of filters, so we have established a semilattice embedding of $\langle A; +\rangle$ into $\langle \mathcal{F}(\omega^N); \cap\rangle$. $\square$

Example 6. If $\mathbf{A}$ is a subdirectly irreducible mode of finite type, then $\mathbf{A}$ may contain a strictly increasing chain of order-type equal to $\omega^N + 1$ where N is the net arity of $\mathsf{V}(\mathbf{A})$. To justify this claim, we shall construct a mode of the form $\mathbf{S}(U)$ which is defined with N binary basic operations and contains an increasing chain of order-type $\omega^N + 1$.

We let the universe of $\mathbf{A}$ be the ordinal $\omega^N + 1$ and for $\alpha, \beta \in A$ define $\alpha + \beta = \max\{\alpha, \beta\}$. $\langle A; +\rangle$ has a least element 0 and a second least element $u \stackrel{\text{def}}{=} 1$. Hence conditions $(a) - (c)$ of Example 1 hold for $\mathbf{S} = \langle A; +\rangle$. Now for $i < N$ and for $\alpha = a_{N-1}\omega^{N-1} + \cdots + a_{i+1}\omega^{i+1} + a_i\omega^i + \cdots + a_0$ define

$$f_i(\alpha) \quad = \begin{cases} a_{N-1}\omega^{N-1} + \cdots + a_{i+1}\omega^{i+1} + (a_i - 1)\omega^i + \cdots + a_0 & \text{if } a_i > 0 \\ a_{N-1}\omega^{N-1} + \cdots + a_{i+1}\omega^{i+1} & \text{if } a_i = 0 \end{cases}$$

and for all i define $f_i(\omega^N) = \omega^N$. In other words, each f_i fixes ω^N while for $\alpha < \omega^N$ f_i reduces the i^{th} coefficient by 1 if that is possible. Otherwise, f_i truncates α at the i^{th} coefficient. Each f_i is a decreasing endomorphism of $\langle A; +\rangle$ and it is easy to check that f_i commutes with f_j for all i, j. Let U be the monoid of endomorphisms of $\langle A; +\rangle$ generated by the f_is. U satisfies $(i) - (iii)$ of Example 1. To see that U satisfies (iv) of Example 1, it suffices to show that if $0 < \alpha < \beta$, then there is an f_i such that $f_i(\alpha) < \min\{\alpha, f_i(\beta)\}$; for then some composition of finitely many of the f_is will be a member of U which maps α to 0 and β to something greater than 0. If $\beta = \omega^N$, then f_{N-1} works. Otherwise we may write $\alpha = a_{N-1}\omega^{N-1} + \cdots + a_0$ and $\beta = b_{N-1}\omega^{N-1} + \cdots + b_0$. Let j be the largest subscript such that $b_j \neq 0$. Then choosing $f_i = f_j$ works. Thus, $\mathbf{S}(U)$ is a subdirectly irreducible mode defined with N binary basic operations which contains an increasing chain of order-type $\omega^N + 1$.

Next, we use $\mathbf{R}(\mathcal{V})$ to determine the lattice of equational theories extending the theory of $\mathcal{V}$.

THEOREM 4.20 *If $\mathcal{V}$ is a variety of semilattice modes, then the lattice of equational theories extending the theory of $\mathcal{V}$ is isomorphic to* $\mathbf{Con}\,\mathbf{R}(\mathcal{V})$.

Proof: By Theorem 3.6 every equational theory T extending the theory of $\mathcal{V}$ has a basis of binary equations relative to $\mathcal{V}$ where each is of the form $r(x,y) = s(x,y)$ with r and s representing elements of $\mathbf{R}(\mathcal{V})$. The collection of all pairs (r,s) where $r(x,y) = s(x,y)$ is in T is clearly a congruence of $\mathbf{R}(\mathcal{V})$.

To finish the proof we must show that if (r,s) and $(p^j, q^j) \in R(\mathcal{V})^2$, $j \in J$, and (r,s) is not in the congruence on $\mathbf{R}(\mathcal{V})$ generated by $\{(p^j, q^j) \mid j \in J\}$, then $r(x,y) = s(x,y)$ is not in the theory generated by the equations $\{p^j(x,y) = q^j(x,y) \mid j \in J\}$ and the theory of $\mathcal{V}$. Let θ be the congruence on $\mathbf{R}(\mathcal{V})$ generated by $\{(p^j, q^j) \mid j \in J\}$. Let T be the collection of equations $t(\bar{x}) = u(\bar{x})$ such that $(\hat{t}_i, \hat{u}_i) \in \theta$ for all i. We claim that T is an equational theory (clearly containing the theory of $\mathcal{V}$). T is certainly an equivalence relation on terms. Using Lemma 3.5 and the fact that θ is a congruence, it is easy to show that T is closed under replacement and substitution. Hence it is an equational theory. Furthermore, each $p^j(x,y) = q^j(x,y)$ is in T. To verify this, using the definition of T, we must show that equations arising from corresponding coefficients belong to T. That is, we must show that $(\hat{p}^j_0, \hat{q}^j_0), (\hat{p}^j_1, \hat{q}^j_1) \in \theta$ for each $j \in J$. But $(\hat{p}^j_0, \hat{q}^j_0) = (p^j, q^j)$ which belongs to θ, since it is one of the defining generators of θ and $(\hat{p}^j_1, \hat{q}^j_1) = (x+y, x+y)$ belongs to θ, since θ is reflexive. Hence, $p^j(x,y) = q^j(x,y)$ is in T. But $r(x,y) = s(x,y)$ is not in T since $(\hat{r}_0, \hat{s}_0) = (r(x,y), s(x,y)) \notin \theta$. □

We will now show that the structure of $\mathbf{R}(\mathcal{V})$ heavily influences the values of the free spectrum function of $\mathcal{V}$. Let $f_{\mathcal{V}}(n)$ equal the cardinality of the free $\mathcal{V}$-algebra on n generators. The function $f_{\mathcal{V}}$ is called the **free spectrum function** of $\mathcal{V}$. $\mathcal{V}$ is locally finite if and only if $f_{\mathcal{V}}(n)$ is finite for $n < \omega$. If $\mathcal{V}$ is locally finite, then we write

$$\mathcal{V}(z) = \sum_{n=0}^{\infty} f_{\mathcal{V}}(n) z^n$$

and we call $\mathcal{V}(z)$ the **generating function for** $f_{\mathcal{V}}$ (or simply the **generating function for** $\mathcal{V}$). (Beware: [10] calls $\frac{1}{z}V(z)$ the generating function for $\mathcal{V}$.) $\mathcal{V}$ is said to be **analytic** if $\mathcal{V}(z)$ converges in some neighborhood of the origin in the complex plane. $\mathcal{V}$ is said to be **rational** if $\mathcal{V}(z)$ is a rational function of z. This means that $\mathcal{V}(z) = P(z)/Q(z)$ for polynomials $P(z)$ and $Q(z)$. Since $\mathcal{V}(z)$ is real and defined at $z = 0$, we may assume that $P(z)$ and $Q(z)$ are real, have no factors in common and $Q(0) \neq 0$. In particular, if $\mathcal{V}(z)$ is rational, then it is analytic. Thus every rational variety is analytic and clearly every analytic variety is locally finite. We are interested in the following problem which appears as Problem 279 in [10]:

Problem: Are the containments relating the classes of locally finite, analytic and rational varieties of modes proper?

We conjecture that the answer is "no"; that in fact every variety of modes is rational. Below we shall prove that a finite join of analytic varieties is analytic. Then, we prove that any locally finite variety of semilattice modes is rational and therefore analytic.

We call a function $f \in \omega^\omega$ **log-linear** if for sufficiently large n we have $f(n) < 2^{cn}$ for some real number c (whence $\log f \in O(n)$).

THEOREM 4.21 *$\mathcal{V}$ is analytic iff $\mathcal{V}$ has log-linear free spectrum.*

Proof: $\mathcal{V}(z)$ is analytic iff it has a positive radius of convergence. Since the values of $f_{\mathcal{V}}(n)$ are positive real numbers, we get that

$$|\mathcal{V}(z)| = |\sum_{n=0}^{\infty} f_{\mathcal{V}}(n)z^n| \leq \sum_{n=0}^{\infty} f_{\mathcal{V}}(n)|z|^n$$

and so $\mathcal{V}(z)$ converges (absolutely) at all z with $|z| \leq r$ if it converges at r. Hence $\mathcal{V}(z)$ is analytic iff $\mathcal{V}(\frac{1}{t})$ converges for some positive real number t. But if $\Sigma_{n=0}^{\infty} f_{\mathcal{V}}(n)\left(\frac{1}{t}\right)^n$ converges, then $\lim_{n\to\infty} f_{\mathcal{V}}(n)\left(\frac{1}{t}\right)^n = 0$. For all sufficiently large n we have $f_{\mathcal{V}}(n)\left(\frac{1}{t}\right)^n < 1$ or just $f_{\mathcal{V}}(n) < t^n = 2^{cn}$ ($c = \log t$). Thus, $f_{\mathcal{V}}$ is log-linear. Conversely, if $f_{\mathcal{V}}(n) < 2^{cn}$. for some c and all large n, then $\mathcal{V}(z)$ converges at the positive real number $2^{-(c+1)}$. To see this, note that

$$\sum_{n=0}^{\infty} |f_{\mathcal{V}}(n)(2^{-(c+1)})^n| = \sum_{n=0}^{\infty}\left(\frac{f_{\mathcal{V}}(n)}{2^{cn}}\right)2^{-n}$$

which converges by comparison with a geometric series. □

COROLLARY 4.22 *The join of two analytic varieties is analytic.*

Proof: If $\mathcal{U}$ and $\mathcal{V}$ are analytic, then $f_{\mathcal{U}}(n) < 2^{cn}$ for large n and $f_{\mathcal{V}}(n) < 2^{dn}$ for large n. Let $\mathcal{W} = \mathcal{U} \vee \mathcal{V}$. Since $\mathbf{F}_{\mathcal{W}}(n)$ is a subdirect product of $\mathbf{F}_{\mathcal{U}}(n)$ and $\mathbf{F}_{\mathcal{V}}(n)$, we get that $f_{\mathcal{W}}(n) \leq f_{\mathcal{U}}(n)f_{\mathcal{V}}(n) < 2^{cn}2^{dn} = 2^{(c+d)n}$ for large values of n. □

Now we proceed with our proof that any variety of semilattice modes is rational. In this proof we shall write $\lambda(n,k)$ to denote the number of surjective functions from an n-element set to a k-element set. A well-known formula for $\lambda(n,k)$ is $S(n,k)k!$ where $S(n,k)$ denotes the Stirling number of the second kind. The number of functions from an n-element set to a k-element set whose range does not include a specified subset of l elements is easily seen to be $(k-l)^n$. Using the principle of inclusion and exclusion we obtain from this the formula

$$\lambda(n,k) = \sum_{i=0}^{k}(-1)^i \binom{k}{i} (k-i)^n$$

which is also well-known. Let $\Lambda_k(z) = \sum_{n=0}^{\infty} \lambda(n,k)z^n$. Using the previously displayed expression we find that

$$\Lambda_k(z) = \sum_{i=0}^{k} \frac{(-1)^i \binom{k}{i}}{(1-(k-i)z)}.$$

For a fixed k, the function $\Lambda_k(z)$ is a finite sum of rational functions so $\Lambda_k(z)$ is rational.

THEOREM 4.23 *Let $\mathcal{V}$ be a locally finite variety of semilattice modes. Then there is a finite sequence of positive natural numbers $(k_0, \ldots, k_{r-1})$ such that $\mathcal{V}(z) = \sum_{i<r} \Lambda_{k_i}(z)$. Hence $\mathcal{V}$ is rational.*

Proof: We must first calculate $f_{\mathcal{V}}(n)$ for a fixed but arbitrarily chosen value of n. Recall that $f_{\mathcal{V}}(n) = |F_{\mathcal{V}}(n)|$ by definition. It is also true that $f_{\mathcal{V}}(n)$ is equal to the number of $\mathcal{V}$-inequivalent n-ary terms. We will use the latter formulation in this proof.

The idea of the proof will be to exploit the coefficient representation for terms. Recall from Lemma 3.5 that every n-ary term has a representation $r_0 \bullet x_0 + \cdots + r_{n-1} \bullet x_{n-1}$ where each r_i may be identified with a member of $\mathbf{R}(\mathcal{V})$. Furthermore, two terms are $\mathcal{V}$-equivalent iff they have the same (i.e., $\mathcal{V}$-equivalent) corresponding coefficients. Hence the number of $\mathcal{V}$-inequivalent terms in n variables equals the number of coefficient representations in the variables $x_0, \ldots, x_{n-1}$ with coefficients from $\mathbf{R}(\mathcal{V})$.

For a given term t of any arity ≥ 1 we define the coefficient set C_t to be the subset $C_t \subseteq R(\mathcal{V})$ which represents the set of distinct (i.e., $\mathcal{V}$-inequivalent) coefficients of t. The collection $C = \{C_t \mid t \text{ a term}\}$ is finite since $\mathbf{R}(\mathcal{V})$ is finite. Now each term $s(x_0, \ldots, x_{n-1})$ **determines** a function from its set of variables $\{x_0, \ldots, x_{n-1}\}$ onto some C_t (in fact, onto C_s). The function it determines is the one that maps a variable to its coefficient in the coefficient representation for s. Clearly $\mathcal{V}$-inequivalent terms will determine different functions because they have different coefficient representations. What is not obvious, but what we shall prove, is that every function from $\{x_0, \ldots, x_{n-1}\}$ onto some C_t is determined by some term in the way just described. For suppose that

$$f : \{x_0, \ldots, x_{n-1}\} \to C_t$$

is onto. Then since C_t is the set of coefficients of some term t, we have

$$t(x_0, \ldots, x_{\ell-1}) = r_0 \bullet x_0 + \cdots + r_{\ell-1} \bullet x_{\ell-1}$$

where C_t is the set of distinct elements in the sequence $(r_0, \ldots, r_{\ell-1})$. If two coefficients are equal in $\mathcal{V}$, for simplicity say $\mathcal{V} \models r_0 = r_1$, then we may use Lemma 3.5 (vii) to re-express t as

$$t(x_0, \ldots, x_{\ell-1}) = r_0 \bullet (x_0 + x_1) + \cdots + r_{\ell-1} \bullet x_{\ell-1}.$$

By grouping together all variables with equal coefficients in this way we get that

$$\begin{aligned} t(\bar{x}) &= u_0 \bullet (\Sigma Y_0) + \cdots + u_{m-1} \bullet (\Sigma Y_{m-1}) \\ &= q(\Sigma Y_0, \ldots, \Sigma Y_{m-1}) \end{aligned}$$

where ΣY_i denotes the semilattice join of members of Y_i and

(i) $C_t = C_q$,

(ii) the coefficients $u_0, \ldots, u_{m-1}$ of q are distinct from each other,

(iii) $\{Y_0, \ldots, Y_{m-1}\}$ is a partition of $\{x_0, \ldots, x_{\ell-1}\}$.

Define $Z_i = f^{-1}(u_i)$. Each Z_i is nonempty, since f is onto $C_t = C_q$. $\{Z_0, \ldots, Z_{m-1}\}$ is a partition of $\{x_0, \ldots, x_{n-1}\}$, since f is a function. Now the term $q(\Sigma Z_0, \ldots, \Sigma Z_{m-1})$ is n-ary and the coefficient of x_i is equal to u_j where j is chosen so that $x_i \in Z_j = f^{-1}(u_j)$. If $v(\bar{x}) = q(\Sigma Z_0, \ldots, \Sigma Z_{m-1})$, then the function determined by v is just f. This concludes the justification that each function from $\{x_0, \ldots, x_{n-1}\}$ onto some C_t is determined by a term.

We now know that the number of $\mathcal{V}$-inequivalent n-ary terms is equal to the number of functions from $\{x_0, \ldots, x_{n-1}\}$ onto some $C_t \in C$. Arranging the numbers $|C_t|$ into a (finite) sequence $(k_0, \dots, k_{r-1})$ gives us that $f_{\mathcal{V}}(n) = \Sigma_{i<r}\lambda(n, k_i)$. The statement of the theorem follows from this. $\square$

The argument we used in Theorem 4.23 shows that a variety $\mathcal{V}$ of semilattice modes is determined up to term equivalence by specifying $\mathbf{R}(\mathcal{V})$ and the coefficient sets $C_t \subseteq R(\mathcal{V})$. For, knowing these things, one can easily describe the clone of $\mathcal{V}$: The n-ary component of the clone may be expressed as the collection of all those coefficient representations in $\{x_0, \dots, x_{n-1}\}$ whose set of coefficients equals some C_t. Composition of terms is described using Lemma 3.5 and the semiring operations. So the question arises as to whether the coefficient sets are directly computable from $\mathbf{R}(\mathcal{V})$. If so, then $\mathcal{V}$ would be recoverable from $\mathbf{R}(\mathcal{V})$. (This possibility is strongly suggested by Corollary 3.7 and Theorem 4.20 and by the analogous result for varieties of Mal'cev modes.) Unfortunately, this is not the case even for locally finite varieties as we show in the next example.

Example 7. In this example we show that it is possible to have varieties of modes $\mathcal{U}$ and $\mathcal{V}$ of the same similarity type where $\mathbf{R}(\mathcal{U}) \cong \mathbf{R}(\mathcal{V})$, but $\mathcal{U}$ and $\mathcal{V}$ have different free spectra. This example arose out of an attempt to find which subsets of $R(\mathcal{V})$ might occur as one of the coefficient sets C_t as in the last proof. My hope was that, using only $\mathbf{R}(\mathcal{V})$, one could algorithmically determine the free spectrum function for $\mathcal{V}$. Moreover, I wanted to show how to reconstruct $\mathcal{V}$ from $\mathbf{R}(\mathcal{V})$. This example shows that this is impossible.

Our varieties will be of type $\langle 2, 3\rangle$ and both satisfy the idempotent and entropic laws. The binary operation will be denoted by $x + y$ for both varieties and it will satisfy the commutative and associative laws as well. The ternary operation for $\mathcal{U}$ will be denoted $p(x, y, z)$ and it will satisfy the laws

$$\hat{p}_0 = \hat{p}_1,\ \hat{p}_1\hat{p}_1 = \hat{p}_1,\ \text{and}\ \hat{p}_2\hat{p}_2 = \hat{p}_2.$$

(Notation borrowed from Lemma 3.5.) The ternary operation for $\mathcal{V}$ will be denoted by $q(x, y, z)$ and $\mathcal{V}$ will satisfy the laws

$$\hat{q}_0 + \hat{q}_1 = 1,\ \hat{q}_2 = 1,\ \hat{q}_0\hat{q}_0 = \hat{q}_0,\ \text{and}\ \hat{q}_1\hat{q}_1 = \hat{q}_1.$$

No other laws except those that are consequences of these hold in $\mathcal{U}$ or $\mathcal{V}$.

$\mathbf{R}(\mathcal{U})$ is generated by the coefficients of its basic operations. If $a = \hat{p}_0$, $b = \hat{p}_1$ and $c = \hat{p}_2$, then $a = b$, $b^2 = b$, $c^2 = c$ and (since $p(x, x, x) = x$) $b + c = 1$. Hence $\mathbf{R}(\mathcal{U})$ is a homomorphic image of the commutative semiring satisfying $1 + r = 1$ which has the presentation $\langle b, c \mid b + c = 1, b^2 = b, c^2 = c\rangle$. This semiring is just the square of the 2-element bounded distributive lattice. There is an "obvious" model of the equations of $\mathcal{U}$ which permits one to verify that in fact $\mathbf{R}(\mathcal{U}) \cong \langle b, c \mid b + c = 1, b^2 = b, c^2 = c\rangle$. That model is the collection of annihilator ideals of $\langle b, c \mid b + c = 1, b^2 = b, c^2 = c\rangle$ with operations $x_0 \oplus x_1 = x_0 \cap x_1$ and $p(x_0, x_1, x_2) = b^{-1}(x_0) \oplus b^{-1}(x_1) \oplus c^{-1}(x_2)$.

Now we use similar arguments for $\mathcal{V}$. If $q(x, y, z) = a \bullet x + b \bullet y + c \bullet z$, then the equations of $\mathcal{V}$ imply that $a + b = 1$, $c = 1$, $a^2 = a$ and $b^2 = b$. Arguing as above we find that

$$\mathbf{R}(\mathcal{V}) \cong \langle b, c \mid b + c = 1, b^2 = b, c^2 = c\rangle \cong \mathbf{R}(\mathcal{U}).$$

Finally we explain why $\mathcal{U}$ and $\mathcal{V}$ have different free spectra. Each basic operation of $\mathcal{V}$ has a coefficient equal to 1. This property is inherited by all terms. Hence the number of $\mathcal{V}$-inequivalent binary terms is the number of expressions of the form (i) $r \bullet x + y$ or (ii) $x + r \bullet y$. Since $|R(\mathcal{V})| = 4$, we get that $f_{\mathcal{V}}(2) = 4 + 4 - 1$. (There are 4 expressions of form (i) and 4 of form (ii) and $x + y$ is of both forms.) However, $\mathcal{U}$ also has 7 binary terms of form (i) or (ii) simply because $|R(\mathcal{U})| = 4$. But $\mathcal{U}$ has other binary terms, e.g. $p(x, x, y) = b \bullet x + c \bullet y$ where $b \neq 1 \neq c$. One calculates in fact that $f_{\mathcal{U}}(2) = 9 > 7 = f_{\mathcal{V}}(2)$.

While Example 7 shows that we cannot generally expect to reconstruct $\mathcal{V}$ from $\mathbf{R}(\mathcal{V})$, there is a special instance where this is possible.

THEOREM 4.24 *Assume that 1 is join-irreducible in $\mathbf{R}(\mathcal{V})$. Then $\mathcal{V}$ may be reconstructed from $\mathbf{R}(\mathcal{V})$. In particular, this can be done whenever $\mathcal{V}$ is generated by a single subdirectly irreducible algebra.*

Proof: By the claim that "$\mathcal{V}$ may be reconstructed from $\mathbf{R}(\mathcal{V})$" we mean that it is possible to construct a variety term equivalent to $\mathcal{V}$ from $\mathbf{R}(\mathcal{V})$. To prove this, it will suffice to prove that a finite subset $C \subseteq R(\mathcal{V})$ is a coefficient set if and only if $1 \in C$. Hence, the coefficient sets are directly computable from $\mathbf{R}(\mathcal{V})$. (The reason that this is sufficient is explained in the paragraph prior to Example 7.) Certainly, if $1 \in C$, then C is a coefficient set. For if $C = \{1, r_0, \ldots, r_k\}$, then for

$$t(x_0, \ldots, x_k, y) = r_0(x_0, y) + \cdots + r_k(x_k, y)$$

is certainly a term, since it involves only $+$ and coefficients, and we have $C_t = C$ (the coefficient of x_i is r_i while the coefficient of y is 1). Conversely, if $C = C_s$ for some s, then the idempotence of s implies that the members of C (i.e., the coefficients of s), must sum/join to 1. (This deduction repeatedly uses Lemma 3.5 (4) to show that the sum of the coefficients of s is equal to the coefficient of the unary term $s(x, \ldots, x) = x$. But this coefficient is easily seen to be 1 by applying the definition in the paragraph preceding Lemma 3.5.) Since 1 is join-irreducible, we must have $1 \in C$. This proves that if 1 is join-irreducible in $\mathbf{R}(\mathcal{V})$, then the coefficient sets are precisely the finite subsets of $R(\mathcal{V})$ which contain 1. This implies the first statement of the theorem.

Next we claim that if $\mathcal{V}$ is generated by a single subdirectly irreducible, then 1 is join-irreducible in $\mathbf{R}(\mathcal{V})$. To see this, we combine Theorems 3.2 and 4.13 to see that (i) the semiring $\langle U; \circ, +, id_A, 0\rangle$ has an annihilator ideal consisting of all non-identity functions in U and (ii) $\mathbf{R}(\mathcal{V}) \cong \langle U; \circ, +, id_A, 0\rangle$. Conclusion (i) is true since the non-identity functions are precisely those that satisfy $p(0) = p(u)$ and this set of functions is easily seen to have the correct closure properties to be an annihilator ideal. From (ii) we get that $R(\mathcal{V}) - \{1\}$ is an annihilator ideal; therefore 1 is join-irreducible. $\square$

In the case where 1 is join-irreducible, the coefficient sets can be computed from $\mathbf{R}(\mathcal{V})$ and so the free spectrum depends only on the integer $r := |R(\mathcal{V})|$. In this case $\mathcal{V}(z) = \frac{1}{(1-rz)} - \frac{1}{(1-(r-1)z)}$.

In the proof of Theorem 4.23 we showed that any term operation t of a semilattice mode variety $\mathcal{V}$ is $\mathcal{V}$-equivalent to a term composed from the join operation and a specialization of t. The specialization of t is the one obtained by setting variables with equal coefficients to the same variable. Hence the clone of $\mathcal{V}$ is generated by $+$ and its terms with distinct coefficients. This means that the clone of a variety of modes is generated by its members of arity $\leq |R(\mathcal{V})|$. One immediately deduces the following result.

PROPOSITION 4.25 *On any finite set there are only finitely many idempotent, entropic clones that contain a semilattice operation.* $\square$

5 The Congruence Extension Property

The following lemma due to Alan Day, [1], is a handy tool for proving that a variety has the congruence extension property.

LEMMA 5.1 (Day's Lemma) *A variety $\mathcal{V}$ has the congruence extension property iff whenever $\mathbf{A} \in \mathcal{V}$ has elements a, b, c and d such that $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$, then $(a, b) \in \mathrm{Cg}^{\mathbf{B}}(c, d)$ for $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b, c, d\})$.* $\square$

LEMMA 5.2 *If $\mathbf{A}$ is a semilattice mode, $c, d \in A$ and $p \in \mathrm{Pol}_1\mathbf{A}$, then*

$$p(c) < p(d) \to (p(c), p(d)) = (q(c), q(d))$$

for some $q \in \mathrm{Pol}_1\mathbf{B}$ where $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{p(c), c, d\})$.

Proof: If we have $c, d \in A$ and $p \in \mathrm{Pol}_1\mathbf{A}$ such that $p(c) < p(d)$, then for any extension $\mathbf{A} \leq \mathbf{A}'$ we also have $c, d \in A'$, $p \in \mathrm{Pol}_1\mathbf{A}'$ and $p(c) < p(d)$. If we obtain the conclusion $(p(c), p(d)) = (q(c), q(d))$ for some $q \in \mathrm{Pol}_1\mathbf{B}$ where $\mathbf{B} = \mathrm{Sg}^{\mathbf{X}}(\{p(c), c, d\})$ by working with $\mathbf{X} = \mathbf{A}'$, then the conclusion holds for $\mathbf{X} = \mathbf{A}$. Therefore, by replacing $\mathbf{A}$ by an extension which is a direct product of subdirectly irreducible algebras, we may assume that $\mathbf{A}$ has a neutral element, 0, for the semilattice operation.

Assume that $p(x) = t^{\mathbf{A}}(x, \bar{u})$ for some term t and some tuple of elements $\bar{u} \in A^n$. Then we have

$$\begin{aligned} p(c) &= t^{\mathbf{A}}(c, \bar{u}) \\ &= t^{\mathbf{A}}(c + 0, 0 + u_0, \ldots, 0 + u_{n-1}) \\ &= t^{\mathbf{A}}(c, 0, \ldots, 0) + t^{\mathbf{A}}(0, \bar{u}). \end{aligned}$$

Let $t_0(x, y) = t(x, y, y, \ldots, y)$, $a = p(c)$ and $v = t^{\mathbf{A}}(0, \bar{u})$. Note that $a = t_0^{\mathbf{A}}(c, 0) + v$, so $v \leq a$. Now

$$\begin{aligned} a &= a + a \\ &= (t_0^{\mathbf{A}}(c, 0) + v) + a \\ &= t_0^{\mathbf{A}}(c, 0) + a \\ &= t_0^{\mathbf{A}}(c, 0) + t_0^{\mathbf{A}}(a, a) \\ &= t_0^{\mathbf{A}}(c + a, a). \end{aligned}$$

Let $q(x) = t_0^{\mathbf{B}}(x + a, a) = t_0^{\mathbf{B}}(x + p(c), p(c)) \in \mathrm{Pol}_1\mathbf{B}$. We have shown that $p(c) = a = t_0^{\mathbf{A}}(c + a, a) = t_0^{\mathbf{B}}(c + a, a) = q(c)$. We now argue that $p(d) = t_0^{\mathbf{A}}(d + a, a) = t_0^{\mathbf{B}}(d + a, a)$

$= q(d)$.

$$\begin{aligned} p(d) &= t^{\mathbf{A}}(d, \bar{u}) \\ &= t^{\mathbf{A}}(d, 0, \ldots, 0) + t^{\mathbf{A}}(0, \bar{u}) \\ &= t_0^{\mathbf{A}}(d, 0) + v \\ &= (t_0^{\mathbf{A}}(d, 0) + v) + a \\ &= t_0^{\mathbf{A}}(d, 0) + a \\ &= t_0^{\mathbf{A}}(d, 0) + t_0^{\mathbf{A}}(a, a) \\ &= t_0^{\mathbf{A}}(d + a, a) \\ &= t_0^{\mathbf{B}}(d + a, a) = q(d). \end{aligned}$$

The only non-obvious step in this argument is the step where we replace $t_0^{\mathbf{A}}(d, 0) + v$ with $(t_0^{\mathbf{A}}(d, 0) + v) + a$. This requires our hypothesis that

$$t_0^{\mathbf{A}}(d, 0) + v = p(d) > p(c) = a.$$

With this comment we conclude the proof. □

THEOREM 5.3 *A variety of semilattice modes has the congruence extension property.*

Proof: By Day's Lemma it suffices to prove that if $\mathbf{A}$ is a semilattice mode containing elements a, b, c and d such that $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$, then $(a, b) \in \mathrm{Cg}^{\mathbf{B}}(c, d)$ where $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b, c, d\})$. So assume that $(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d)$. By Mal'cev's congruence generation theorem we can find a sequence of elements $a = x_0, x_1, \ldots, x_n = b$ such that for all $i < n$ we have $\{x_i, x_{i+1}\} = \{p_i(c), p_i(d)\}$ for some $p_i \in \mathrm{Pol}_1\mathbf{A}$. We alter this to a new sequence

$$a = y_0, \ldots, y_n = z_0, \ldots, z_n = b$$

by defining $y_i = x_0 + \cdots + x_i$ for $i \leq n$ and $z_j = x_j + \cdots + x_n$ for $j \leq n$. We further define $r_i(x) = p_i(x) + y_i \in \mathrm{Pol}_1\mathbf{A}$ and $s_i(x) = p_i(x) + z_{i+1} \in \mathrm{Pol}_1\mathbf{A}$ for $i < n$. The sequence of polynomials

$$(r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1})$$

witnesses the fact that

$$a = y_0, \ldots, y_n = z_0, \ldots, z_n = b$$

is a Mal'cev chain connecting a to b by polynomial images of $\{c, d\}$. This chain has the further property that $y_i \leq y_{i+1}$ and $z_{i+1} \leq z_i$. By deleting unnecessary links in the chain if necessary, we may assume that these inequalities are strict.

Either $r_0(c) = y_0 < y_1 = r_0(d)$ or $r_0(d) = y_0 < y_1 = r_0(c)$. In either case, Lemma 5.2 proves that there is a unary polynomial q_0 of the algebra $\mathrm{Sg}^{\mathbf{A}}(\{c, d, y_0\})$ such that $(r_0(c), r_0(d)) = (q_0(c), q_0(d))$. But as $y_0 = a$, this means that q_0 is the restriction of a polynomial of $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b, c, d\})$. Hence $y_1 \in \{q_0(c), q_0(d)\} \subseteq B$. Now we can repeat this argument with $y_1 < y_2$ in place of $y_0 < y_1$ and get a polynomial $q_1 \in \mathrm{Pol}_1\mathbf{B}$ such that $(r_1(c), r_1(d)) = (q_1(c), q_1(d))$ and deduce that, for $\mathbf{B}' = \mathrm{Sg}^{\mathbf{A}}(\{y_1, b, c, d\})$,

$$y_2 \in \{q_1(c), q_1(d)\} \subseteq B' \subseteq B.$$

In this way we can prove that for all $i \leq n$ we have $y_i \in B$ and that for $i < n$ there is a $q_i \in \mathrm{Pol}_1\mathbf{B}$ such that $(q_i(c), q_i(d)) = (r_i(c), r_i(d))$. Similar arguments prove that for all $i \leq n$ we have $z_i \in B$ and that for $i < n$ there is a $t_i \in \mathrm{Pol}_1\mathbf{B}$ such that $(t_i(c), t_i(d)) = (s_i(c), s_i(d))$. Hence

$$a = y_0, \ldots, y_n = z_0, \ldots, z_n = b$$

is a Mal'cev chain witnessing the fact that $(a, b) \in \mathrm{Cg}^{\mathbf{B}}(c, d)$. This finishes the proof. $\square$

6 Comments and Problems

We have learned that most questions about a variety of semilattice modes can be reduced to questions about the associated semiring and then usually handled quite easily. The questions about semilattice modes that remain naturally divide themselves into two classes. First, how much more needs to be said about reducing semilattice mode problems to problems concerning only the associated semiring? Second, how much more needs to be understood about the structure of the associated semiring? Beyond this we may ask how much of what we know about semilattice modes is true for all mode varieties or for varieties with a compatible semilattice operation. We shall list some specific problems of each type which remain open.

The most important question concerning the reduction of semilattice mode problems to problems concerning only the associated semiring is the following.

Question 1. Is there an easy way to determine which finite subsets of $\mathbf{R}(\mathcal{V})$ are the set of coefficients of some term?

For example, if one knows the 2-element coefficient sets, then can one calculate all the coefficient sets? We consider Question 1 to be important because $\mathcal{V}$ can be recovered from $\mathbf{R}(\mathcal{V})$ and the collection of coefficient sets. This means that, in a sense, we have a complete reduction of any problem about $\mathcal{V}$ to a problem concerning only $\mathbf{R}(\mathcal{V})$ and the collection of coefficient sets. How little about the coefficient sets do we need to know to recover $\mathcal{V}$?

Now we ask about further properties of $\mathbf{R}(\mathcal{V})$.

Question 2. If $\mathbf{R}$ is a finitely generated, commutative semiring satisfying $1 + r = 1$, then does $\mathbf{R}$ satisfy the ascending chain condition on congruences?

We conjecture that the answer is yes. From a proof of this conjecture one would obtain a new proof of Corollary 4.4 since the lattice of annihilator ideals is isomorphic to the sublattice of $\mathbf{Con\ R}$ consisting of ideal congruences. More importantly, a positive answer to Question 4 can be combined with Theorem 4.20 to prove that any variety of semilattice modes of finite type is finitely based.

Finally we list some questions about extending some of the results here to varieties related to semilattice mode varieties.

Question 3. Is every variety of modes residually small? Is every variety of modes of finite type residually countable?

Question 4. If $\mathcal{V}$ is a variety of modes, is $\mathcal{V} = \mathsf{V}(\mathbf{F}_{\mathcal{V}}(2))$?

Question 5. Is every mode variety axiomatized by the entropic laws and binary equations?

Many of the results here have been proved by viewing semilattice modes as semimodules over $\mathbf{R}(\mathcal{V})$. The same type of argument is possible for varieties of affine modes. Is there some common ground?

Question 6. Let $\mathcal{V}$ be a variety of modes having a ternary operation $p(x, y, z)$ which satisfies every regular equation true of the operation $x - y + z$ in the variety of abelian groups. Then can one construct from $p(x, y, z)$ and binary terms a semiring which determines most of the properties of $\mathcal{V}$?

Varieties of affine modes have an abelian group term $p(x, y, z) = x-y+z$ while varieties of semilattice modes have the ternary join $p(x, y, z) = x + y + z$ which satisfies every regular equation. A locally finite variety of modes satisfies the Mal'cev condition of Question 8 if and only if it contains no strongly solvable algebras.

Recently, R. McKenzie has shown that if $\mathbf{A}$ is finite and $\mathsf{V}(\mathbf{A})$ is a residually small variety, then any subdirectly irreducible algebra $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ which has nonabelian monolith must have a tolerance θ with $\leq |A|$ blocks such that polynomial operations of $\mathbf{B}$ restricted to each θ-class are compatible with some semilattice operation (expect this in [7]). This shows that understanding the structure of algebras in residually small varieties will require first understanding the structure of algebras which have a compatible semilattice operation. In McKenzie's proof, the semilattice operation is not necessarily given by a term. Hence we suggest:

Problem 7. Describe the varieties of algebras where each member has a compatible semilattice operation. (What if the semilattice operation is given by a term?)

McKenzie's discovery has a striking analogy with the known properties of congruence modular varieties. If $\mathbf{A}$ is finite, $\mathsf{V}(\mathbf{A})$ is congruence modular and residually small, then $\mathsf{V}(\mathbf{A})$ will not contain large subdirectly irreducibles with nonabelian monolith. But there may be large subdirectly irreducibles in $\mathsf{V}(\mathbf{A})$ with abelian monolith. If $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ is one, then $\mathbf{B}$ has a large congruence θ with $\leq |A|$ classes such that the polynomial operations of $\mathbf{B}$ restricted to each θ-class are compatible with abelian group operations. (The author has extended this result to non-modular varieties if "abelian monolith" is replaced by "abelian, but not strongly abelian, monolith.") Those familiar with congruence modular varieties will recognize that the results discussed in this paragraph are a part of "commutator theory". McKenzie's new result suggests that there may be a different type of commutator for non-modular varieties where the analogue of an affine algebra is an algebra which has a compatible semilattice term. We are led to ask:

Question 8. Is it possible to develop a theory analogous to modular commutator theory for a large class of varieties where the analogue of an affine algebra is an algebra with a compatible semilattice term?

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