Keith A. Kearnes

Quasivarieties of Modules Over Path Algebras of Quivers

Dedicated to the memory of Willem Johannes Blok

Abstract. Let $\mathbb{F}\Gamma$ be a finite dimensional path algebra of a quiver Γ over a field $\mathbb{F}$. Let $\mathcal{L}$ and $\mathcal{R}$ denote the varieties of all left and right $\mathbb{F}\Gamma$-modules respectively. We prove the equivalence of the following statements.

- The subvariety lattice of $\mathcal{L}$ is a sublattice of the subquasivariety lattice of $\mathcal{L}$.
- The subquasivariety lattice of $\mathcal{R}$ is distributive.
- Γ is an ordered forest.

Keywords: Quasivariety, quiver, path algebra, distributive lattice.

1. Introduction

Let $\mathcal{L}$ be the variety of all left modules over a ring $\mathbf{R}$. Call the subvariety lattice of $\mathcal{L}$ its V*-lattice*, and the subquasivariety lattice of $\mathcal{L}$ its Q*-lattice*. This paper addresses two questions raised in [1] concerning the following properties.

A. The V-lattice of $\mathcal{L}$ is a sublattice of the Q-lattice of $\mathcal{L}$.

B. The Q-lattice of $\mathcal{L}$ is distributive.

The questions from [1] are:

I. What structural properties of $\mathbf{R}$ guarantee that Property A holds?

II. Does Property A imply Property B?

The V-lattice of any congruence modular variety is modular, and the Q-lattice of any quasivariety is join semidistributive. Since a modular sublattice of a join semidistributive lattice is distributive, it follows that if Property A holds for the congruence modular variety $\mathcal{L}$, then the V-lattice of $\mathcal{L}$ must be distributive. This shows that Property A implies a weak form of Property B,

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hence one might say that Question II has a weak affirmative answer for any ring $\mathbf{R}$.

Nevertheless, there exist rings for which the answer to Question II is negative. In this paper, we show that if $\mathbb{F}\Gamma$ is a finite dimensional path algebra over a quiver, then the variety of left $\mathbb{F}\Gamma$-modules has Property A if and only if Γ is an ordered forest (thereby answering Question I for this class of rings). We then show that the variety of left $\mathbb{F}\Gamma$-modules has Property B if and only if Γ^{op} is an ordered forest. Since the class of ordered forests is not self dual, Properties A and B are independent of one another for varieties of left modules over path algebras of quivers.

Although the purpose of this paper is to supply a negative answer to Question II, our result may be viewed as supplying an affirmative answer to a noncommutative version of this question (for a certain class of rings). This is interesting, since it appears that the authors of [1] formulated Question II based on evidence from [3, 4] pertaining to quasivarieties of modules over commutative rings. Our results show that if $\mathbf{R}$ is a finite dimensional path algebra over a quiver, then Property A for the variety of *left* $\mathbf{R}$-modules is equivalent to Property B for the variety of *right* $\mathbf{R}$-modules.

2. Quivers and Their Representations

Informally, a 'quiver' is a directed multigraph. We formalize this and fix language in the following definition.

Definition 2.1. A *quiver* is a 2-sorted structure $\Gamma = \langle V, E; h, t\rangle$ where the elements of sort V are called the *vertices* of Γ, the elements of sort E are called the *edges* of Γ, and $h, t\colon E \to V$ are functions called *head* and *tail*. Γ is *finite* if $V \cup E$ is. (We shall always assume that $V \cap E = \emptyset$.)

A *source* is a vertex $u \in V$ such that $h^{-1}(u) = \emptyset$; a *sink* is a vertex $v \in V$ such that $t^{-1}(u) = \emptyset$.

A *nontrivial path* in Γ is a finite sequence $\pi = e_n \cdots e_2 e_1$, $n \geq 1$, of elements of E such that $h(e_i) = t(e_{i+1})$ for every i. This path π *starts* at $t(e_1)$ and *ends* at $h(e_n)$. For each $v \in V$ the *trivial path at* v, denoted π_v, is defined to be v itself. This path starts and ends at v. A *path* is any trivial or nontrivial path.

A *directed cycle* is a nontrivial path that starts and ends at the same vertex. A quiver without directed cycles is *acyclic*.

Paths π_1 and π_2 are *composable* if π_2 starts where π_1 ends. If π_1 and π_2 are composable nontrivial paths, then their *composition* is the concatenation $\pi_2\pi_1$ of the two paths. If π starts at v, then the composition of π_v and π is

$\pi\pi_v := \pi$, while if π ends at v, then the composition of π and π_v is $\pi_v\pi := \pi$. The composition of π_v with itself is π_v.

The quiver *opposite* to $\Gamma = \langle V, E; h, t\rangle$ is the quiver $\Gamma^{op} = \langle V, E; t, h\rangle$ obtained by interchanging h and t.

In fact, a quiver is a partial description of a small category (with objects V, morphisms E, and domain and codomain maps t and h respectively), and Γ^{op} is a partial description of the opposite category. A 'representation' of a quiver Γ is defined to correspond to a functor from the category freely generated by Γ to the category of vector spaces over some field, and a 'homomorphism' of representations is defined to correspond to a natural transformation between such functors.

Definition 2.2. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-*representation* of Γ is a function ρ with domain $V \cup E$ such that

(a) $\rho(v)$ is an $\mathbb{F}$-vector space for each $v \in V$, and
(b) $\rho(e)$ is a linear transformation from $\rho(t(e))$ to $\rho(h(e))$ for each $e \in E$.

The *dimension* of a representation ρ is $\dim_{\mathbb{F}}(\oplus_{v\in V}\rho(v))$.

A *homomorphism* $\varphi\colon \rho_1 \to \rho_2$ between representations is a set $\{\varphi_v \mid v \in V\}$ of linear transformations $\varphi_v\colon \rho_1(v) \to \rho_2(v)$ such that

$$\rho_2(e) \circ \varphi_u = \varphi_v \circ \rho_1(e) \tag{i}$$

whenever $e \in E$ and $(t(e), h(e)) = (u, v)$.

The assignment $e \mapsto \rho(e)$ of a linear transformation to each edge in (b) above extends to paths: $\pi := e_n \cdots e_2e_1 \mapsto \rho(e_n) \circ \cdots \circ \rho(e_2) \circ \rho(e_1) =: \rho(\pi)$ and $\pi_v \mapsto id_{\rho(v)}$.

Given the homomorphism concept, the definitions of *subrepresentation*, *quotient*, *product* and *(direct) sum* are evident.

Definition 2.3. Let $\mathbb{F}$ be a field. The *path algebra* of Γ over $\mathbb{F}$, denoted $\mathbb{F}\Gamma$, is the associative $\mathbb{F}$-algebra whose underlying $\mathbb{F}$-space has a basis consisting of the paths of Γ and whose multiplication is defined on paths by

$$\pi_2 \cdot \pi_1 = \begin{cases} \pi_2\pi_1 & \text{if } \pi_1 \text{ and } \pi_2 \text{ are composable} \\ 0 & \text{otherwise.} \end{cases}$$

The following facts will be used.

Theorem 2.4. (1) $\mathbb{F}\Gamma$ *is a unital* $\mathbb{F}$-*algebra if and only if* Γ *has finitely many vertices. (In which case,* $1 = \sum_{v\in V}\pi_v$.*)*

(2) $\mathbb{F}\Gamma$ *is finite dimensional if and only if* Γ *is finite and acyclic. (See Proposition III.1.1 of [2].)*
(3) *If* Γ *has finitely many vertices, the category of* $\mathbb{F}$*-representations of* Γ *and their homomorphisms is equivalent to the category of left* $\mathbb{F}\Gamma$*-modules and their homomorphisms. (See Theorem III.1.5 of [2] for the equivalence between finite dimensional representations and modules.)*

We sketch the constructions that give the equivalence in item (3). If A is an $\mathbb{F}\Gamma$-module, then define an $\mathbb{F}$-representation of Γ by letting $\rho_A(v) = \pi_v A$ and, if $e \in E$ and $(t(e), h(e)) = (u, v)$, letting $\rho_A(e)\colon \pi_u A \to \pi_v A$ be left multiplication by the one-edge path e. If $\varphi\colon A \to B$ is an $\mathbb{F}\Gamma$-homomorphism, then $\{\varphi|_{\pi_v A} \mid v \in V\}$ is the corresponding homomorphism $\rho_A \to \rho_B$.

For the other direction, if ρ is an $\mathbb{F}$-representation of Γ, define $A = \oplus_{v\in V}\rho(v)$. Let $p_v\colon \oplus_{v\in V}\ \rho(v) \to \rho(v)$ and $i_v\colon \rho(v) \to \oplus_{v\in V}\rho(v)$ be the canonical projections and injections, $v \in V$. Make the space A a left $\mathbb{F}\Gamma$-module by defining left scalar multiplication by a path π that starts at u and ends at v to be the linear transformation $i_v \circ \rho(\pi) \circ p_u$. If $\varphi = \{\varphi_v \mid v \in V\}$ is a homomorphism from ρ_A to ρ_B, then the function $\oplus_{v\in V}\varphi_v$ acting diagonally on $A = \oplus_{v\in V}\rho_A(v)$ is the corresponding homomorphism from A to B.

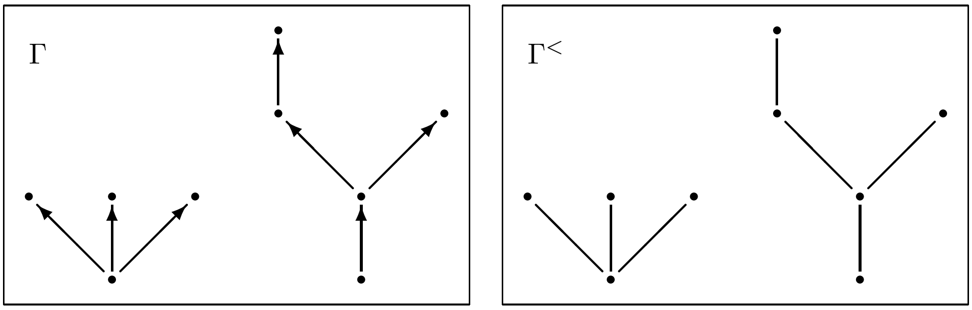
Now we introduce a key concept for the results of this paper.

Definition 2.5. A finite acyclic quiver $\Gamma = \langle V, E; h, t\rangle$ is an *ordered forest* if h is injective.

To explain this terminology, let $E^\circ = \{(t(e), h(e)) \mid e \in E\}$. The acyclicity of Γ forces the transitive closure of E° to be a strict partial order $<$ on V. Denote the partially ordered set $\langle V; <\rangle$ by $\Gamma^<$. Since h is injective, Γ may be recovered from $\Gamma^<$ by taking $E = \{(u, v) \mid v \text{ covers } u \text{ in } \Gamma^<\}$, $t\colon E \to V\colon (u, v) \mapsto u$ and $h\colon E \to V\colon (u, v) \mapsto v$. Thus, Γ and $\Gamma^<$ determine one another. Now, in the language of ordered sets, the injectivity of h translates into the statement that any principal order ideal in $\Gamma^<$ is a chain. (For, if $a, b < c$ in $\Gamma^<$ and a and b are incomparable, then there is a first vertex d along any given path from a to c in Γ that is also on some path from b to c. But if this happens then $|h^{-1}(d)| \geq 2$, contradicting the injectivity of h.) Therefore the connected components of $\Gamma^<$ are trees, i.e. $\Gamma^<$ is a forest.

3. Subdirectly irreducible representations

A module is *subdirectly irreducible* if it has a nonzero submodule, called its *monolith*, that is contained in all nonzero submodules. There is a correspond-



AN ORDERED FOREST, AND ITS ASSOCIATED PARTIALLY ORDERED SET

ing concept for representations. The subdirectly irreducible representations of a finite acyclic quiver are described in this section.

DEFINITION 3.1. Let ρ be a representation of a quiver $\Gamma = \langle V, E; h, t\rangle$. The *support* of ρ is the subquiver $\Gamma|_\rho := \langle V', E'; h', t'\rangle$ where $V' = \{v \in V \mid \rho(v) \neq \{0\}\}$, $E' = \{e \in E \mid \rho(e) \neq 0\}$, $h' = h|_{E'}$, and $t' = t|_{E'}$.

THEOREM 3.2. *A representation ρ of a finite acyclic quiver Γ is subdirectly irreducible if and only if*

(a) $\Gamma|_\rho$ *has a unique sink* $v_0 \in V$*,*
(b) $\dim(\rho(v_0)) = 1$*, and*
(c) *For each* $v \in V - \{v_0\}$*, the set of functions*

$$\{\rho(\pi) \mid \pi \textit{ is a nontrivial path from } v \textit{ to } v_0.\}$$

separates the points of $\rho(v)$*.*

PROOF. Assume that ρ is subdirectly irreducible. Then ρ is not the zero representation, so $\Gamma|_\rho$ is not empty. Since Γ is finite and acyclic, $\Gamma|_\rho$ must have a sink, $v_0 \in V$. Let U_0 be a 1-dimensional subspace of $\rho(v_0)$ and let U be an arbitrary 1-dimensional subspace of $\rho(v)$, v a sink of $\Gamma|_\rho$. The subrepresentations generated y U and U_0 are 1-dimensional. If $U \neq U_0$, then these subrepresentations of ρ are disjoint, contradicting the subdirect irreducibility of ρ. Thus $U = U_0$, establishing that (a) and (b) hold.

If there is a vertex $v \in V - \{v_0\}$ and vectors $a, b \in \rho(v)$ such that $\rho(\pi)(a) = \rho(\pi)(b)$ for all paths starting at v and ending at v_0, then the 1-dimensional subrepresentation of ρ generated by $a - b \in \rho(v)$ is disjoint from the 1-dimensional representation generated by $\rho(v_0)$, again contradicting subdirect irreducibility. Thus (c) holds.

Now suppose that (a)–(c) hold. Let σ be the nonzero subrepresentation of ρ whose support has vertex set $\{v_0\}$. (There is exactly one, by (a) and (b).) If ρ' is any subrepresentation of ρ of positive dimension, and $a \in \rho'(v)$ is nonzero for some $v \in V$, then it follows from (c) that there is a path π starting at v and ending at v_0 such that $\rho'(\pi)(v) \neq 0$. By this conclusion and (b), $1 \leq \dim_{\mathbb{F}}(\rho'(v_0)) \leq \dim_{\mathbb{F}}(\rho(v_0)) = 1$. By the choice of σ, $\dim_{\mathbb{F}}(\sigma(v_0)) = 1$. Hence $\rho'(v_0) = \sigma(v_0)$. Since σ is supported only at v_0, σ is a subrepresentation of ρ'. Since $\rho' \leq \rho$ was chosen arbitrarily, σ is the least nonzero subrepresentation of ρ, and ρ is subdirectly irreducible. ■

COROLLARY 3.3. *Let Γ be an ordered forest. A representation ρ of Γ is subdirectly irreducible if and only if*

(a) *the vertex set of $\Gamma|_\rho$ is a (linearly ordered) interval in $\Gamma^<$,*
(b) $\dim(\rho(v)) = 1$ *for all vertices in $\Gamma|_\rho$, and*
(c) $\rho(e)$ *is an isomorphism for all edges in $\Gamma|_\rho$.*

PROOF. Assume that ρ is subdirectly irreducible, and that $v_0 \in V$ is its sink. If $v \neq v_0$ is another vertex in $\Gamma|_\rho$, then Theorem 3.2 (c) implies that there is a directed path from v to v_0. Hence the vertices in $\Gamma|_\rho$ lie in the order ideal of $\Gamma^<$ that is generated by v_0. This ideal is linearly ordered since Γ is an ordered forest. Call it C.

For each vertex v of $\Gamma|_\rho$ there is a unique path π of Γ that starts at v and ends at v_0. (Uniqueness follows from the injectivity of h, existence follows from Theorem 3.2 (c).) Since $\rho(\pi)$ is a the unique linear transformation from $\rho(v)$ to $\rho(v_0)$ in the point-separating set of transformations guaranteed by Theorem 3.2 (c), $\rho(\pi)$ is injective. Since $\rho(v_0)$ is 1-dimensional, each $\rho(v)$ must also be 1-dimensional (item (b)). If $v < u < v_0$ in $\Gamma^<$, then the unique path π from v to v_0 must pass through u. Since $\rho(\pi)$ is an isomorphism, u must belong to the support of ρ. Hence the vertex set of $\Gamma|_\rho$ is a filter in $\langle C; < \rangle$, equivalently it is an interval in $\Gamma^<$ (item (a)). Since $\rho(\pi)$ is an isomorphism for each path in $\Gamma|_\rho$ that ends at v_0 and all edges in $\Gamma|_\rho$ belong to such a path, it follows that $\rho(e)$ is an isomorphism for all e in $\Gamma|_\rho$ (item (c)).

If (a)–(c) of this corollary hold, then (a)–(c) of Theorem 3.2 hold, hence ρ is subdirectly irreducible. ■

The following consequence of Corollary 3.3 will be needed in Section 4.

COROLLARY 3.4. *Let Γ be an ordered forest. If M and N are subdirectly irreducible left $\mathbb{F}\Gamma$-modules, then M and N have isomorphic monoliths if and only if M is embeddable in N or N is embeddable in M.*

Proof. Since the monoliths of each are the unique minimal submodules, any embedding of one into the other must restrict to an isomorphism between monoliths.

For the converse, let ρ_M and ρ_N be the corresponding representations. Their supporting vertex sets are linearly ordered intervals in $\Gamma^<$, say C and D. The sinks of $\Gamma|_{\rho_M}$ and $\Gamma|_{\rho_N}$ are the maximal elements u_0 and v_0 of C and D respectively. According to the proof of Theorem 3.2, the monoliths of M and N correspond to the subrepresentations supported by $\{u_0\}$ and $\{v_0\}$ respectively. The scalar $\pi_{u_0} \in \mathbb{F}\Gamma$ acts like the identity on the monolith of M, and like zero on the monolith of N unless $u_0 = v_0$. If M and N have isomorphic monoliths, then necessarily $u_0 = v_0$. The order ideal of $\Gamma^<$ generated by this element contains both C and D as upper intervals, hence one of C or D is a contained in the other as an order filter. Assume that

$$C = \{v_0 > v_1 > \cdots > v_m\} \subseteq \{v_0 > v_1 > \cdots > v_n\} = D.$$

Under this assumption, an embedding of M into N may be constructed by defining $\varphi_0 \colon \rho_M(v_0) \to \rho_N(v_0)$ to be an arbitrary $\mathbb{F}$-space isomorphism, which must exist since both are 1-dimensional, and then defining linear transformations $\varphi_i \colon \rho_M(v_i) \to \rho_N(v_i)$ inductively by $\varphi_{i+1} = \rho_N^{-1}(e_i) \circ \varphi_i \circ \rho_M(e_i)$ where $e_i \in E$ is the unique edge from v_{i+1} to v_i. That $\varphi = \{\varphi_i \mid 1 \leq i \leq m\}$ is a homomorphism follows immediately from (i), while the fact that it is an embedding follows from the fact that each of the transformations $\varphi_0, \rho_M(e_i)$ and $\rho_N(e_i)$ is an isomorphism. ∎

4. Property A

If $\mathbf{R}$ is a ring and $\mathcal{L}$ is the variety of left $\mathbf{R}$-modules, then Property A is the property that the V-lattice of $\mathcal{L}$ is a sublattice of the Q-lattice of $\mathcal{L}$. In this section we shall prove that when $\mathbf{R} = \mathbb{F}\Gamma$ is the path algebra of a finite acyclic quiver, then $\mathbf{R}$ has Property A if and only if Γ is an ordered forest.

Throughout this section and the next we use the following notation. If $\mathcal{K}$ is a class of similar algebras, then $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$, $\mathbf{P}(\mathcal{K})$, and $\mathbf{P}_u(\mathcal{K})$ denote the closure of the class under the formation of homomorphic images, subalgebras, products and ultraproducts respectively. Each of these closure operators is assumed to include closure under isomorphisms.

Lemma 4.1. Let $\mathcal{U}$ be a variety. The V-lattice of $\mathcal{U}$ is a sublattice of the $\mathcal{Q}$-lattice of $\mathcal{U}$ if and only if the variety generated by a subdirectly irreducible algebra of $\mathcal{U}$ is join prime in the V-lattice of $\mathcal{U}$.

Proof. The equivalence of consecutive statements on the following list is clear, while the equivalence of the negations of the first and last is what is to be proved.

(a) The V-lattice of $\mathcal{U}$ is not a sublattice of the Q-lattice.

(b) There exist subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ whose V-lattice join is larger than their Q-lattice join.

(c) Their exist subvarieties $\mathcal{V}, \mathcal{W}$ and a subdirectly irreducible algebra $A \in \mathbf{HSP}(\mathcal{V} \cup \mathcal{W}) - \mathbf{SP}(\mathcal{V} \cup \mathcal{W})$.

(d) There is a subdirectly irreducible algebra $A \in \mathcal{U}$ and subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ such that $\mathbf{HSP}(A) \not\leq \mathcal{V}$, $\mathbf{HSP}(A) \not\leq \mathcal{W}$, but $\mathbf{HSP}(A) \leq \mathbf{HSP}(\mathcal{V} \cup \mathcal{W})$. ■

Theorem 4.2. *If the variety of left* $\mathbb{F}\Gamma$*-modules satisfies Property A, then* Γ *is an ordered forest.*

Proof. Let $\mathcal{L}$ be the variety of left $\mathbb{F}\Gamma$-modules. We will argue that if Γ is not an ordered forest, then $\mathcal{L}$ does not satisfy Property A. According to Definition 2.5 and Lemma 4.1, our task is to derive from the noninjectivity of the head operation of Γ the existence of a subdirectly irreducible $\mathbb{F}\Gamma$-module that generates a subvariety of $\mathcal{L}$ that is not join prime in the V-lattice of $\mathcal{L}$. So assume that Γ has distinct edges d and e with equal heads ($h(d) = h(e)$). We separate the argument into two cases depending on whether or not the tails of these edges are equal.

Case 1. $t(d) \neq t(e)$.

Define a representation ρ_A of Γ by taking $\rho_A(v)$ be a copy of the 1-dimensional space $\mathbb{F}$ if the vertex v is at the head or tail of d or e ($v \in \{h(d) = h(e), t(d), t(e)\}$), and taking $\rho_A(v) = \{0\}$ otherwise. Define $\rho_A(d)$ and $\rho_A(e)$ be $id_{\mathbb{F}}$ and $\rho_A(f) = 0$ for all other edges $f \in E$. The support of ρ_A has vertex set $\{h(d) = h(e), t(d), t(e)\}$ and edge set $\{d, e\}$.

$\Gamma|_{\rho_A}$ ρ_A

Let ρ_M be the subrepresentation of ρ_A for which $\rho_M(t(d)) = \rho_M(h(d)) = \mathbb{F}$ and $\rho_M(t(e)) = \{0\}$, let ρ_N be the subrepresentation for which $\rho_N(t(e)) =$

$\rho_N(h(e)) = \mathbb{F}$ and $\rho_N(t(d)) = \{0\}$, and let ρ_P be their intersection (so $\rho_P(t(d)) = \rho_P(t(e)) = \{0\}$ and $\rho_P(h(d)) = \mathbb{F}$).

$$\begin{tikzcd} & \mathbb{F} & \\ \mathbb{F} \arrow[ur, "id_{\mathbb{F}}"] & & \{0\} \arrow[ul, "0_{\mathbb{F}}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{F} & \\ \{0\} \arrow[ur, "0_{\mathbb{F}}"] & & \mathbb{F} \arrow[ul, "id_{\mathbb{F}}"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{F} & \\ \{0\} \arrow[ur, "0_{\mathbb{F}}"] & & \{0\} \arrow[ul, "0_{\mathbb{F}}"'] \end{tikzcd}$$

ρ_M $\qquad\qquad$ ρ_N $\qquad\qquad$ ρ_P

Let M, N and P be the associated submodules of A. It follows from the criteria of Theorem 3.2 that A, M, N and P are all subdirectly irreducible. They are related by $P \subseteq M \subseteq A$ and $P \subseteq N \subseteq A$. Since $\dim_{\mathbb{F}}(P) = 1$, $\dim_{\mathbb{F}}(M) = \dim_{\mathbb{F}}(N) = 2$, and $\dim_{\mathbb{F}}(A) = 3$, P is the least nonzero submodule of each of them, and $Q := M/P, R := N/P$ and P are simple modules. No two of these simple modules are isomorphic, since if $r_P := \pi_{h(d)}, r_Q := \pi_{t(d)}$ and $r_R := \pi_{t(e)}$ then the scalar r_X acts like the identity on the simple module X and like zero on the modules in $\{P, Q, R\} - \{X\}$.

The variety generated by M does not contain A, since the composition factors of the finite dimensional members of $\mathbf{HSP}(M)$ must be among the composition factors of M, which are P and Q, and the module $R \notin \{P, Q\}$ occurs as a composition factor of A. Similarly, the variety generated by N does not contain A. Thus, A is a subdirectly irreducible $\mathbb{F}\Gamma$-module satisfying $\mathbf{HSP}(A) \not\leq \mathbf{HSP}(M)$ and $\mathbf{HSP}(A) \not\leq \mathbf{HSP}(N)$. We complete the proof of Claim 1 by showing that $\mathbf{HSP}(A) \leq \mathbf{HSP}(M, N)$. Indeed, the inclusion maps $i_M \colon M \to A$ and $i_N \colon N \to A$ induce a map of the coproduct $i_M \sqcup i_N \colon M \oplus N \to A$ whose image contains the images of i_M and i_N, hence contains $i_M(M) \cup i_N(N) = M \cup N$. Since M and N are distinct 2-dimensional submodules of A, $i_M \sqcup i_N(M \oplus N)$ contains a 3-dimensional submodule of A. But A is 3-dimensional, so $i_M \sqcup i_N$ maps $M \oplus N$ onto A. Since $M \oplus N = M \times N \in \mathbf{HSP}(M, N)$, this puts A in $\mathbf{HSP}(M, N)$.

Case 2. $t(d) = t(e)$.

In this case, d and e are edges from $u := t(d) = t(e)$ to $v := h(d) = h(e)$. Let ρ_A, ρ_M and ρ_N be representations of Γ whose supporting vertex set in each case is $\{d, e\}$. Define $\rho_A(u) = \rho_A(v) = \rho_M(u) = \rho_M(v) = \rho_N(u) = \rho_N(v) = \mathbb{F}$, considered as a 1-dimensional space. Define $\rho_A(d) = \rho_A(e) = \rho_M(d) = \rho_N(e) = id_{\mathbb{F}}$, and $\rho_M(e) = \rho_N(d) = 0_{\mathbb{F}}$. Let A, M and N be the associated modules.

The $\mathbb{F}\Gamma$-module identity $e \cdot x = 0$ holds in M but not A, and $d \cdot x = 0$ holds in N but not A. Thus, $\mathbf{HSP}(A) \not\leq \mathbf{HSP}(M)$ and $\mathbf{HSP}(A) \not\leq \mathbf{HSP}(N)$. We now argue that $\mathbf{HSP}(A) \leq \mathbf{HSP}(M, N)$.

Let ρ_L be the subrepresentation $\rho_M \oplus \rho_N$ generated by the vector $g := (1_{\mathbb{F}}, 1_{\mathbb{F}}) \in \rho_M(u) \oplus \rho_N(u)$. The space $\rho_L(u)$ equals $\mathbb{F} \cdot g$, while the space $\rho_L(v)$ is spanned by $\rho_L(d)(g) = (1_{\mathbb{F}}, 0_{\mathbb{F}}) \in \rho_L(v)$ and $\rho_L(e)(g) = (0_{\mathbb{F}}, 1_{\mathbb{F}}) \in \rho_L(v)$. Define a homomorphism $\varphi\colon \rho_L \to \rho_A$ by $\varphi_u\colon \rho_L(u) \to \rho_A(u)\colon g \mapsto 1_{\mathbb{F}}$,

$$\varphi_v\colon \rho_L(v) \to \rho_A(v)\colon (1_{\mathbb{F}}, 0_{\mathbb{F}}) \mapsto 1_{\mathbb{F}}, (0_{\mathbb{F}}, 1_{\mathbb{F}}) \mapsto 1_{\mathbb{F}},$$

and $\varphi_w = 0$ for $w \notin \{u, v\}$. To verify that φ is a homomorphism, it suffices to show that $\rho_A(d) \circ \varphi_u(g) = \varphi_v \circ \rho_L(d)(g)$ and $\rho_A(e) \circ \varphi_u(g) = \varphi_v \circ \rho_L(e)(g)$. In both cases, each side reduces to $1_{\mathbb{F}} \in \rho_A(v)$.

The homomorphism φ is surjective, since $\varphi_u(g) = 1_{\mathbb{F}} \in \rho_A(u)$ is a generator for A. Thus, ρ_A is a quotient of the subrepresentation $\rho_L \leq \rho_M \oplus \rho_N$, implying that $A \in \mathbf{HSP}(M, N)$. ■

Next we prove the converse of Theorem 4.2.

THEOREM 4.3. *If* Γ *is an ordered forest, then the variety* $\mathcal{L}$ *of left* $\mathbb{F}\Gamma$*-modules satisfies Property A.*

PROOF. We must argue that every subdirectly irreducible left $\mathbb{F}\Gamma$-module generates a join prime subvariety of the variety $\mathcal{L}$ of all left $\mathbb{F}\Gamma$-modules. Assume instead that $A \in \mathcal{L}$ is subdirectly irreducible, $A \in \mathbf{HSP}(\mathcal{V} \cup \mathcal{W})$ for certain subvarieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{L}$, but $A \notin \mathcal{V} \cup \mathcal{W}$. Since $\mathbf{HSP}(\mathcal{V} \cup \mathcal{W}) = \mathbf{HS}(\{B \times C \mid B \in \mathcal{V}, C \in \mathcal{W}\})$, there must exist modules $B \in \mathcal{V}, C \in \mathcal{W}$, and submodules $D \leq E \leq B \times C$ such that $E/D \cong A$. By replacing B and C with submodules if necessary, we may assume that the inclusion $E \subseteq B \times C$ is a subdirect representation.

Since $E/D \cong A$, there is a module D^* covering D in the submodule lattice of E, such that D^*/D is the monolith of E/D. Extend D to a submodule $D' \leq B \times C$ that is maximal for $D^* \nsubseteq D'$. Then $D \subseteq E \cap D'$ but $D^* \nsubseteq E \cap D'$, so $E \cap D' = D$ (since D^* is the least submodule of E that properly contains D). By the maximality of D', $(B \times C)/D'$ is subdirectly irreducible with monolith $(D^* + D')/D'$. By the second isomorphism theorem,

$$A \cong E/D = E/(E \cap D') \cong (E + D')/D' \subseteq (B \times C)/D'.$$

Thus, $A' := (B \times C)/D' \in \mathbf{HSP}(\mathcal{V} \cup \mathcal{W})$ is a subdirectly irreducible module contained in the same join of varieties as A, and $A' \notin \mathcal{V} \cup \mathcal{W}$ since it contains an isomorphic copy of A as a submodule. Since A' satisfies all the properties required of A, we may replace A by A', in which case D gets replaced by

D' and E gets replaced by $B \times C$. Changing notation back, we may assume that $E = B \times C$, hence $A = (B \times C)/D$.

Choose submodules $B' \leq B$ and $C' \leq C$ maximal for the property that $B' \times C' \subseteq D$. By replacing B, C, D with $B/B', C/C'$, and $D/(B' \times C')$, we may further assume that $B \times C$ has no nonzero product submodule contained in D. In particular, since $(B \times \{0\}) \cap D$ and $(\{0\} \times C) \cap D$ are product submodules contained in D, they are zero. Thus,

$$\begin{aligned} B &\cong (B \times \{0\})/\{0\} = (B \times \{0\})/((B \times \{0\}) \cap D) \\ &\cong ((B \times \{0\}) + D)/D \leq (B \times C)/D \cong A, \end{aligned}$$

forcing B (and similarly C) to be isomorphic to a nonzero submodule of A. This forces B and C to be subdirectly irreducible, and to have monoliths isomorphic to the monolith of A.

Since B and C have isomorphic monoliths, Corollary 3.4 guarantees that either B is embeddable in C or C is embeddable in B. Assuming the former, we get that $B \in \mathbf{HSP}(C) \subseteq \mathcal{W}$, so $A \in \mathbf{HSP}(B, C) \subseteq \mathcal{W}$, contrary to our choice of A. ■

Remark 4.4. We have shown that if $\mathbf{R} = \mathbb{F}\Gamma$ is a finite dimensional path algebra of a quiver and $\mathcal{L}$ is the variety of left $\mathbf{R}$-modules, then the following properties are related by the implications (a)⇒(b)⇒(c)⇒(a).

(a) The V-lattice of $\mathcal{L}$ is a sublattice of the Q-lattice.
(b) Γ is an ordered forest.
(c) If M and N are subdirectly irreducible left $\mathbf{R}$-modules with isomorphic monoliths, then one is embeddable in the other.

Our proof of (c)⇒(a) holds for any ring $\mathbf{R}$.

5. Property B

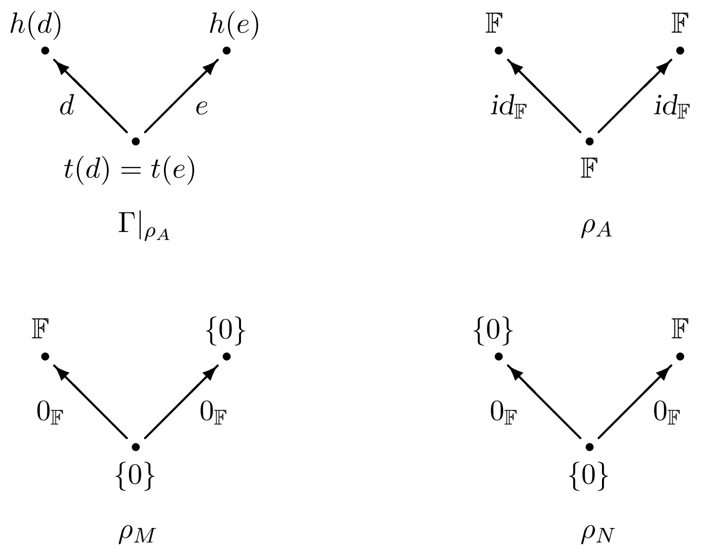
Recall that Property B is the property that the Q-lattice of $\mathcal{L}$ is distributive. In this section we shall prove that $\mathbb{F}\Gamma$ has Property B if and only if Γ^{op} is an ordered forest.

Theorem 5.1. *If the variety of left* $\mathbb{F}\Gamma$*-modules satisfies Property B, then* Γ^{op} *is an ordered forest.*

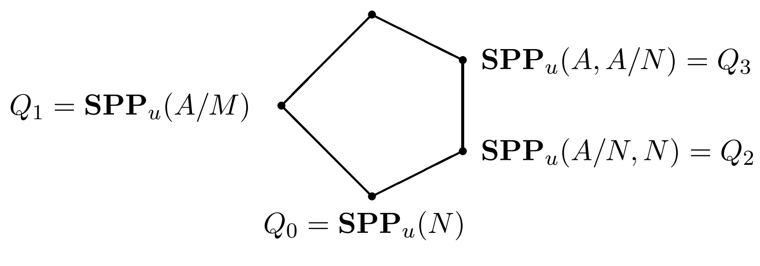
Proof. Γ^{op} is an ordered forest if and only if its tail operation is injective. As in the proof of Theorem 4.2, we will prove $t(d) = t(e)$ implies $d = e$ by considering the cases $h(d) \neq h(e)$ and $h(d) = h(e)$ separately.

Case 1. $h(d) \neq h(e)$.

Let ρ_A be the representation whose three supporting vertices are $t(d) = t(e), h(d)$ and $h(e)$, and which has $\rho_A(t(d)) = \rho_A(h(d)) = \rho_A(h(e)) = \mathbb{F}$ and $\rho_A(d) = \rho_A(d) = id_{\mathbb{F}}$. The associated module A is 3-dimensional. A has 1-dimensional (simple) submodules M and N which are associated to the subrepresentations ρ_M and ρ_N generated by $1_{\mathbb{F}} \in \rho_A(h(d))$ and $1_{\mathbb{F}} \in \rho_A(h(e))$ respectively.



Our goal is to prove that the following nonmodular (hence nondistributive) lattice is a sublattice of the Q-lattice of the variety of all left $\mathbb{F}\Gamma$-modules.

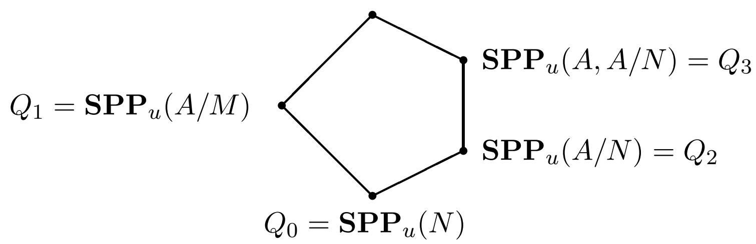


The inclusion $Q_2 \subseteq Q_3$ follows from the fact that N is a submodule of A. The identity $e \cdot x = 0$ is satisfied by A/N and by N, but not by A. This shows that the inclusion is proper. Since the submodule $M \leq A$ is disjoint from N, N is isomorphic to a submodule of A/M; this guarantees that $Q_0 \subseteq Q_1$. The natural map from A to $A/M \times A/N$ is an embedding, since $M \cap N = \{0\}$, so $Q_3 \leq \mathbf{SP}(Q_1 \cup Q_2)$. These facts are enough to show that the joins in this figure are correct. What remains to show is that $Q_1 \cap Q_3 \subseteq Q_0$.

The supporting vertices of A/M are $\{t(e), h(e)\}$, so Q_1 satisfies the identity $\pi_v \cdot x = 0$ for all vertices v not in this set. The quasi-identity $e \cdot x = 0 \longrightarrow x = 0$ expressing the injectivity of the e-map holds in A/M, hence in Q_1. This shows that if $L \in Q_1$, then its supporting vertex set is a subset of $\{t(e), h(e)\}$, and $\rho_L(e)$ is injective. The quasivariety Q_3 satisfies the quasi-identity $d \cdot x = 0 \longrightarrow x = 0$, which expresses the injectivity of the d-map. Thus, if $L \in Q_1 \cap Q_3$, then $\rho_L(d)$ and $\rho_L(e)$ are injective, and the vertices of $\Gamma|_{\rho_L}$ are a subset of $\{t(e), h(e)\}$. But if the d-map is injective and $h(d)$ is not a supporting vertex, then $t(d)$ cannot be a supporting vertex. Hence the only possible supporting vertex of $L \in Q_1 \cap Q_3$ is $h(e)$. This means precisely that the left $\mathbb{F}\Gamma$-module L is annihilated by left multiplication by any path except $\pi_{h(e)}$, which acts like the identity on L. Any two nontrivial modules of this type are embeddable in powers of one another, since this property holds for nontrivial $\mathbb{F}$-vector spaces, and N is one such module. Thus, $(L \in Q_1 \cap Q_3) \Rightarrow (L \leq N^\kappa$ for some $\kappa) \Rightarrow (L \in Q_0)$.

Case 2. $h(d) = h(e)$.

The argument here is similar to the one for Case 1. Let $u := t(d) = t(e)$ and $v := h(d) = h(e)$. Let ρ_A be the 3-dimensional representation of Γ for which $\rho_A(u) = \mathbb{F}, \rho_A(v) = \mathbb{F}^2$ and $\rho_A(w) = \{0\}$ for all other vertices w. Let $\rho_A(d)(x) = (x, 0) \in \mathbb{F}^2$, $\rho_A(e)(x) = (0, x) \in \mathbb{F}^2$, and $\rho_A(f)(x) = 0$ for all other edges f. Let ρ_M be the 1-dimensional subrepresentation of ρ_A whose supporting vertex is v and whose space at this vertex is $\mathbb{F} \cdot (1, 0)$. Let ρ_N be the 1-dimensional subrepresentation whose supporting vertex is v and whose space at this vertex is $\mathbb{F} \cdot (0, 1)$. Let A, M and N be the associated modules. We will prove that the following nonmodular lattice is a sublattice of the Q-lattice of the variety of all left $\mathbb{F}\Gamma$-modules.



It is easy to see that $M \cong N$, since (a) both are 1-dimensional as $\mathbb{F}$-spaces and (b) all paths in $\mathbb{F}\Gamma$ act like zero on each of them, except for the trivial path at v which acts like the identity on both of them. Since M and N are disjoint, $M \cong N \cong (M+N)/M \leq A/M$ and $N \cong M \cong (M+N)/N \leq A/N$, so N is embeddable in both A/M and A/N. The fact that M and N are disjoint also implies that $A \leq A/M \times A/N$, so $Q_3 \leq \mathbf{SP}(Q_1 \cup Q_2)$. These facts show that the joins are correct in the previous figure. It remains to show that $Q_2 \neq Q_3$ and $Q_1 \cap Q_3 \subseteq Q_0$.

To see that $Q_2 \neq Q_3$, note that A/N (and hence Q_2) satisfies the identity $e \cdot x = 0$ while A (and hence Q_3) does not. To see that $Q_1 \cap Q_3 \subseteq Q_0$ choose $L \in Q_1 \cap Q_3$ arbitrarily. The quasivariety $Q_1 = \mathbf{SPP}_u(A/M)$ satisfies the identities $\pi_w \cdot x = 0$ for all vertices $w \notin \{u, v\}$, since the supporting vertices of $\rho_{A/M}$ are $\{u, v\}$. Q_1 satisfies the identity $d \cdot x = 0$ and the quasi-identity $e \cdot x = 0 \longrightarrow x = 0$, since $\rho_{A/M}(d)$ is the zero map and $\rho_{A/M}(e)$ is injective. These identities and quasi-identity are satisfied by L, so the supporting vertices of L are in $\{u, v\}$, $\rho_L(d) = 0$, and $\rho_L(e)$ is injective.

Both $\rho_A(d)$ and $\rho_{A/N}(d)$ are injective, so Q_3 satisfies $d \cdot x = 0 \longrightarrow x = 0$. Hence $L \in Q_3$ satisfies this quasi-identity. Since we have already established that $\rho_L(d)$ is zero, it follows that u is not a supporting vertex of ρ_L. As an $\mathbb{F}\Gamma$-module, L is simply an $\mathbb{F}$-vector space on which each path acts like zero except for the trivial path at v, which acts like the identity. Any nontrivial module of this type is embeddable in a power of any other nontrivial module of this type, and N is a nontrivial module of this type. This shows that $(L \in Q_1 \cap Q_3) \Rightarrow (L \leq N^\kappa$ for some $\kappa) \Rightarrow (L \in Q_0)$. ■

Lemma 5.2. If Γ^{op} is an ordered forest, then any finite dimensional indecomposable left $\mathbb{F}\Gamma$-module is subdirectly irreducible.

Proof. We assume that a finite dimensional $\mathbb{F}\Gamma$-module A is not subdirectly irreducible, and prove that it has a nontrivial direct decomposition.

If A is not subdirectly irreducible, then (since it is finite dimensional) it must have disjoint nontrivial submodules M and N. By extending them if necessary we may assume that if $M' \supseteq M$ and $N' \supseteq N$ are submodules satisfying $M' \cap N' = \emptyset$, then $M' = M$ and $N' = N$. Under this assumption, we will prove that $A = M + N = M \oplus N$. Converting from modules A, M and N to representations ρ_A, ρ_M and ρ_N, our aim is to prove that $\rho_A(v) = \rho_M(v) \oplus \rho_N(v)$ for every $v \in V$. Since M and N are disjoint, it is enough to prove that $\rho_A(v) = \rho_M(v) + \rho_N(v)$ for every $v \in V$.

Assume that this is not so, and let $v_0 \in V$ be a vertex that is maximal in $\Gamma^<$ for the property that $\rho_A(v) \neq \rho_M(v) + \rho_N(v)$. There is at most one edge $e \in E$ such that $t(e) = v_0$, since t is injective when Γ^{op} is an odered forest. If such an edge exists, let v_1 denote $h(e)$. If v_1 exists, then $v_0 < v_1$

in $\Gamma^<$, so the maximality of v_0 guarantees that $\rho_A(v_1) = \rho_M(v_1) + \rho_N(v_1)$. In particular, this means that

$$\rho_A(v_0) = [\rho_A(e)]^{-1}(\rho_A(v_1)) = [\rho_A(e)]^{-1}(\rho_M(v_1)) + [\rho_A(e)]^{-1}(\rho_N(v_1)). \quad \text{(ii)}$$

Now we define new subrepresentations $\rho_{M'}$ and $\rho_{N'}$ of ρ_A extending ρ_M and ρ_N respectively. Define $\rho_{M'}(v) = \rho_M(v)$ and $\rho_{N'}(v) = \rho_N(v)$ for all $v \neq v_0$. Define $\rho_{M'}(v_0) = U$ and $\rho_{N'}(v_0) = W$ where U and W are complementary subspaces of $\rho_A(v_0)$ satisfying

(a) $\rho_M(v_0) \subseteq U \subseteq [\rho_A(e)]^{-1}(\rho_M(v_1))$, and
(b) $\rho_N(v_0) \subseteq W \subseteq [\rho_A(e)]^{-1}(\rho_N(v_1))$.

(The conditions referring to e are in effect only if e exists.) Observe that the leftmost subspaces in (a) and (b) are disjoint (since M and N are), the leftmost subspaces in (a) and (b) are contained in the rightmost subspaces (since ρ_M and ρ_N are subrepresentations of ρ_A), and the rightmost subspaces in (a) and (b) sum to $\rho_A(v_0)$ (by (ii)). It follows that there do exist complementary subspaces U and W satisfying (a) and (b).

The inclusion $\rho_M(v_0) \subseteq U$ guarantees that for any edge $f \in E$ with $h(f) = v_0$ we have $\rho_A(f)(\rho_M(t(f))) \subseteq \rho_M(h(f)) = \rho_M(v_0) \subseteq U$, so $\rho_A(f)$ may be considered to be a linear transformation from $\rho_{M'}(t(f)) := \rho_M(t(f))$ to U. Similarly $\rho_N(v_0) \subseteq W$ guarantees that $\rho_A(f)$ may be considered to be a linear transformation from $\rho_{N'}(t(f)) := \rho_N(t(f))$ to W. Moreover, the inclusions $U \subseteq [\rho_A(e)]^{-1}(\rho_M(v_1))$ and $W \subseteq [\rho_A(e)]^{-1}(\rho_M(v_1))$ guarantee that $\rho_A(e) \colon U \to \rho_M(v_1)$ and $\rho_A(e) \colon W \to \rho_N(v_1)$ are linear transformations. Thus $\rho_{M'}$ and $\rho_{N'}$ are representations. By the choices made, the associated modules M' and N' are disjoint and properly extend M and N. This contradicts the maximality assumption on (M, N), so $A = M + N = M \oplus N$. ■

Theorem 5.3. *If Γ^{op} is an ordered forest, then the variety of left $\mathbb{F}\Gamma$-modules satisfies Property B.*

Proof. We must prove that if Γ^{op} is an ordered forest, then the subquasivariety lattice of the variety $\mathcal{L}$ of all left $\mathbb{F}\Gamma$-modules is distributive.

It suffices to prove that the subquasivariety lattice is modular, since any subquasivariety lattice is join semidistributive and any modular join semidistributive lattice is distributive. Therefore we must show that $Q_0 \cap (Q_1 \vee Q_2) \subseteq (Q_0 \cap Q_1) \vee Q_2$ whenever $Q_0 \supseteq Q_2$. If this is not the case, then there is an $\mathbb{F}\Gamma$-module

$$A \in [Q_0 \cap (Q_1 \vee Q_2)] - [(Q_0 \cap Q_1) \vee Q_2]. \quad \text{(iii)}$$

A may be taken to be finitely generated, hence finite dimensional. By choosing A of least dimension among modules satisfying (iii) we guarantee that A is indecomposable. According to Lemma 5.2, A is subdirectly irreducible. ¿From $A \in Q_0 \cap (Q_1 \vee Q_2)$ we get $A \in Q_0$ and $A \in Q_1 \vee Q_2 = \mathbf{SP}(Q_1 \cup Q_2)$. A is subdirectly irreducible, so the last part of this is equivalent to

$$A \in Q_1 \cup Q_2. \qquad \text{(iv)}$$

If $A \in Q_1$, then since $A \in Q_0$ as well we get $A \in Q_0 \cap Q_1 \subseteq (Q_0 \cap Q_1) \vee Q_2$, which contradicts (iii). On the other hand, if in (iv) we have $A \in Q_2$, then we also contradict (iii). The only conclusion is that there is no module in $[Q_0 \cap (Q_1 \vee Q_2)] - [(Q_0 \cap Q_1) \vee Q_2]$. ■

Remark 5.4. We have shown that if $\mathbf{R} = \mathbb{F}\Gamma$ is a finite dimensional path algebra of a quiver and $\mathcal{L}$ is the variety of left $\mathbf{R}$-modules, then the following properties are related by the implications (a)⇒(b)⇒(c)⇒(a).

(a) The Q-lattice of $\mathcal{L}$ is distributive.

(b) Γ^{op} is an ordered forest.

(c) Every finitely generated, indecomposable, left $\mathbf{R}$-module is subdirectly irreducible.

Our proof of (c)⇒(a) holds for any ring $\mathbf{R}$.

We summarize our results in the form that appears in the abstract.

Corollary 5.5. *Let $\mathcal{L}$ denote the variety of all left $\mathbb{F}\Gamma$-modules, and let $\mathcal{R}$ denote the variety of all right $\mathbb{F}\Gamma$-modules. The following are equivalent.*

(1) *The subvariety lattice of $\mathcal{L}$ is a sublattice of the subquasivariety lattice of $\mathcal{L}$.*

(2) *The subquasivariety lattice of $\mathcal{R}$ is distributive.*

(3) Γ *is an ordered forest.*

Proof. The equivalence of (1) and (3) follows from Theorems 4.2 and 4.3.

It follows from Definition 2.3 that the $\mathbb{F}$-algebra that is opposite to $\mathbb{F}\Gamma$ is the path algebra over the opposite quiver, i.e. $(\mathbb{F}\Gamma)^{op} = \mathbb{F}(\Gamma^{op}) =: \mathbb{F}\Gamma^{op}$. Therefore the variety of left $\mathbb{F}\Gamma^{op}$-modules is equivalent to the variety $\mathcal{R}$ of right $\mathbb{F}\Gamma$-modules. Hence the equivalence of (2) and (3) follows from Theorems 5.1 and 5.3. ■

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Keith A. Kearnes
Department of Mathematics
University of Colorado
Boulder, CO 80309-0395
USA
kearnes@euclid.colorado.edu