CLONES OF ALGEBRAS WITH PARALLELOGRAM TERMS

KEITH A. KEARNES AND ÁGNES SZENDREI

Abstract. We describe a manageable set of relations that generates the finitary relational clone of an algebra with a parallelogram term. This result applies to any algebra with a Maltsev term and to any algebra with a near unanimity term. One consequence of the main result is that on any finite set and for any finite k there are only finitely many clones of algebras with a k-ary parallelogram term which generate residually small varieties.

1. Introduction

A *clone* on a set A is a collection of finitary operations on A that contains the projection operations and is closed under composition. If $\mathbf{A} = \langle A; F_0, F_1, F_2, \ldots\rangle$ is an algebra of some signature, then the *clone of* $\mathbf{A}$ consists of the term operations of $\mathbf{A}$.

A finitary operation f on a set A is *compatible* with a finitary relation $R \subseteq A^n$ (or R is compatible with f) if R is closed under the coordinatewise application of f, equivalently if R is a subuniverse of $\langle A; f\rangle^n$. The set of compatible pairs $(f, R) \in \mathrm{Op}(A) \times \mathrm{Rel}(A)$ of operations and relations defines a Galois connection, $\perp\colon \mathrm{Op}(A) \to \mathrm{Rel}(A)$ and $\perp\colon \mathrm{Rel}(A) \to \mathrm{Op}(A)$, between the set of operations and the set of relations on A. It is shown in [2, 6] that a set $\mathcal{C}$ of operations on a finite set A is a clone if and only if it is closed with respect to this Galois connection, meaning that $\mathcal{C}^{\perp\perp} = \mathcal{C}$. Thus, each clone $\mathcal{C}$ on a finite set A corresponds to a unique Galois closed set of relations $\mathcal{C}^{\perp}$, and for this reason a Galois closed set of relations has come to be called a *(finitary) relational clone*.

To study clones on infinite sets A via this type of Galois connection it is necessary to make some modifications. The natural approach is to let $\mathrm{Op}(A)$ remain the set of finitary operations on A, but let $\mathrm{Rel}(A)$ be the set of all relations of arity $\leq |A|$ (or the class of all relations on A). In this approach the Galois closed sets of operations are again the clones on A. Unfortunately, the corresponding relational clones are usually unmanageable. Alternatively, one may leave the Galois connection of the

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second paragraph unchanged (with both $\mathrm{Op}(A)$ and $\mathrm{Rel}(A)$ finitary), in which case the Galois closed sets of operations are no longer the clones on A, but the "locally closed" clones on A. Here if $\mathcal{C}$ is a clone on A, then $f\colon A^n \to A$ is a *$\mathcal{C}$-local operation* if for each finite subset $U \subseteq A^n$ there is a $g_U \in \mathcal{C}$ such that f and g_U agree on U. The *local closure* of $\mathcal{C}$ is the clone $\overline{\mathcal{C}}$ of all $\mathcal{C}$-local operations, and $\mathcal{C}$ is *locally closed* if $\overline{\mathcal{C}} = \mathcal{C}$. In this approach, one studies clones only up to local closure. One is compensated for the loss of scope by the facts that finitary relational clones are easier to deal with and that many interesting clones are locally closed (e.g. any clone on a finite set or any clone of a free algebra).

It is shown in [9, 10, 12] that a set $\mathcal{R}$ of finitary relations on a possibly infinite set A is a finitary relational clone if and only if it contains the equality relation and is closed under the operations of finite direct product, arbitrary intersection of relations of the same arity, permutation of coordinates, projection onto a subset of coordinates, and directed union of relations of the same arity. Given a finitary relational clone $\mathcal{R}$ it is a basic problem of clone theory to describe a manageable set of relations $\mathcal{G}$ which generates $\mathcal{R}$ under these operations. (Equivalently $\mathcal{G} \subseteq \mathcal{R}$ and $\mathcal{G}^{\perp\perp} = \mathcal{R}$.) To describe such a set it is reasonable to start with $\mathcal{G}_0 = \mathcal{R}$ and then discard unnecessary relations. Since the k-ary relations in $\mathcal{R}$ ordered by inclusion form an algebraic lattice, each k-ary member of $\mathcal{R}$ is the intersection of completely $\cap$-irreducible k-ary members of $\mathcal{R}$. Hence if $\mathcal{G}_1$ is the collection of all completely $\cap$-irreducible members of $\mathcal{R}$, then the fact that relational clones are closed under intersection implies that $\mathcal{G}_1^{\perp\perp} = \mathcal{G}_0^{\perp\perp} = \mathcal{R}$. Next, suppose that $R \in \mathcal{G}_1$ is directly decomposable, say $R = S \times T$ (after possibly permuting coordinates). Using projection onto subsets of coordinates we get $S, T \in \mathcal{R}$. Since direct product distributes over arbitrary intersection, the fact that R is completely $\cap$-irreducible implies that both S and T are completely $\cap$-irreducible. Thus, each indecomposable direct factor of R is simultaneously directly indecomposable and completely $\cap$-irreducible, and belongs to $\mathcal{R}$. Since R is generated by these factors under direct product, if $\mathcal{G}_2$ is defined to be the set of directly indecomposable, completely $\cap$-irreducible relations in $\mathcal{R}$, then $\mathcal{G}_2^{\perp\perp} = \mathcal{G}_1^{\perp\perp} = \mathcal{G}_0^{\perp\perp} = \mathcal{R}$.

These reflections motivate us to call a member of a finitary relational clone *critical* if it is directly indecomposable and completely $\cap$-irreducible. We have just explained why every finitary relational clone is generated by its critical members. The purpose of this paper is to give a structure theorem for certain critical compatible relations of certain algebras. Here if $\mathbf{A}$ is an algebra, then a *critical relation of* $\mathbf{A}$ is a critical relation in the finitary relational clone of compatible relations of $\mathbf{A}$. The assumption that we make about our algebras is that they generate congruence modular varieties, and the assumption that we make about the critical relations we investigate is that they satisfy a parallelogram property (to be defined later). We then define "parallelogram terms" which enforce the parallelogram property for critical relations. Thus, if a variety $\mathcal{V}$ has parallelogram terms, then our results provide valuable insight into the structure of the clones of algebras in $\mathcal{V}$. For example, we will show that on any finite set there are only finitely many clones of algebras that generate residually small varieties and have k-ary parallelogram terms for a fixed k.

The class of varieties with a parallelogram term is definable by a Maltsev condition. This Maltsev condition is stronger than the one defining the class of congruence modular varieties, but weaker than the one defining the class of congruence permutable varieties and also weaker than the one defining the class of varieties with a near unanimity term. In fact, the class of varieties with a $(k+3)$-ary parallelogram term is the same as the class of varieties with a $(k+1)$-ary "edge term" or a (2^k-1)-ary "cube term", both concepts from [1].

The results obtained here extend our results from [7] about clones of finite groups, and also extend the results announced in [13] about clones of finite Maltsev algebras.

2. Critical Relations in Congruence Modular Varieties

In *Kongruenzklassengeometrien*, [14], R. Wille associates to an algebra $\mathbf{A}$ a geometry whose points are the elements of A and whose lines are the classes of congruences on $\mathbf{A}$. Wille's *Parallelogrammaxiom* is the assertion that whenever $p, q, r \in A$ are related by congruences θ and ψ, as in Figure 1,

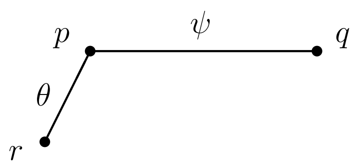

Figure 1.

there is a fourth point $s \in A$ completing the parallelogram, as in Figure 2.

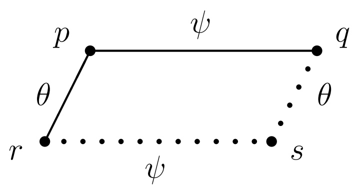

Figure 2.

If $\mathbf{A}$ is an algebra, then we define a *parallelogram* in $\mathbf{A}^k$ to be a subset $\{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}\} \subseteq A^k$ of elements related by congruences $\eta, \eta' \in \operatorname{Con}(\mathbf{A}^k)$ that are disjoint kernels of projections of $\mathbf{A}^k$ onto subsets of coordinates, as in Figure 3.

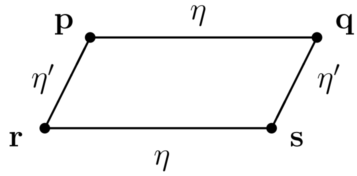

FIGURE 3.

If $k = \{0, \ldots, k-1\}$, η is the kernel of the projection onto the coordinates in $U \subseteq k$ and η' is the kernel of the projection onto the coordinates in $V \subseteq k$, then the assumption that η and η' are disjoint means that $U \cup V = k$. After permuting the coordinates of the tuples in Figure 3 so that those in U are followed by those in $V - U$, the statement that $\{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}\}$ is a parallelogram means that we can factor these tuples as $\mathbf{p} = \mathbf{ac}$, $\mathbf{q} = \mathbf{ad}$, $\mathbf{r} = \mathbf{bc}$, and $\mathbf{s} = \mathbf{bd}$ using tuples whose lengths satisfy $|\mathbf{a}| = |\mathbf{b}| = |U|$ and $|\mathbf{c}| = |\mathbf{d}| = |V - U|$.

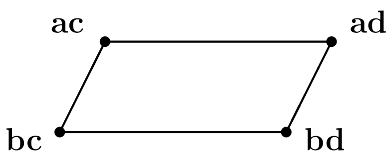

FIGURE 4.

When we need more information we will call a parallelogram an (m, n)-*parallelogram* if $|U| = m$ and $|V - U| = n$ (e.g., if $|\mathbf{a}| = m$ and $|\mathbf{c}| = n$ in Figure 4). We will say that a relation $R \subseteq A^k$ satisfies the *parallelogram property* (or (m, n)-*parallelogram property*) if whenever it contains three vertices of a parallelogram ((m, n)-parallelogram), then it also contains the fourth. This means exactly that whenever $\eta \in \operatorname{Con}(\mathbf{A}^k)$ is the kernel of a projection onto m coordinates and η' is the kernel of the projection onto the complementary set of coordinates, then the restrictions of η and η' to R are permuting equivalence relations.

In this section we will prove a structure theorem for critical relations R which satisfy the following restrictions:

(1) R is a compatible relation of an algebra $\mathbf{A}$ in a congruence modular variety,
(2) R satisfies the $(1, k-1)$-parallelogram property.

We start with an elementary characterization of critical relations. If $\mathbf{s} \in A^k$, then an i-*approximation* of $\mathbf{s}$ is a tuple $\mathbf{t} \in A^k$ such that $s_j = t_j$ for all $j \neq i$ (i.e., $\mathbf{t}$ agrees with $\mathbf{s}$ in all coordinates except possibly the i-th).

Lemma 2.1. *Let $\mathcal{R}$ be a finitary relational clone on A and let $R \in \mathcal{R}$ be k-ary for some $k > 1$. R is critical if and only if there is a tuple $\mathbf{s} \in A^k$ such that*

(1) *R is maximal in $\mathcal{R}$ for the property that $\mathbf{s} \notin R$, and*
(2) *R contains i-approximations of $\mathbf{s}$ for all i.*

Moreover, if R is critical and R^* is the unique upper cover of R in the $\cap$-semilattice of k-ary relations of $\mathcal{R}$, then every tuple $\mathbf{s} \in R^* - R$ has properties (1) *and* (2)*.*

Proof. [$\Rightarrow$, and the last claim of the theorem] Assume that R is critical and R^* is the unique upper cover of R in the $\cap$-semilattice of k-ary relations of $\mathcal{R}$. Choose any tuple $\mathbf{s} \in R^* - R$. Already item (1) holds. Item (2) will follow from the next claim.

Claim 2.2. *If $U \subsetneq k$ is a proper subset of k and $\mathrm{pr}_U \colon A^k \to A^U$ is projection onto the coordinates in U, then $\mathrm{pr}_U(R) = \mathrm{pr}_U(R^*)$.*

If one applies a permutation of coordinates to a critical relation one obtains another critical relation, so there is no harm in assuming that $U = \ell = \{0, \ldots, \ell-1\}$ for some $\ell < k$. Let $V = k - U$. The relation $S := \mathrm{pr}_U(R) \times A^V$ contains R and is directly decomposable, so S must contain R^*. Thus $\mathrm{pr}_U(R) \subseteq \mathrm{pr}_U(R^*) \subseteq \mathrm{pr}_U(S) = \mathrm{pr}_U(R)$, forcing $\mathrm{pr}_U(R) = \mathrm{pr}_U(R^*)$.

To derive item (2) from Claim 2.2 let $U = k - \{i\}$. From $\mathbf{s} \in R^*$ we get $\mathrm{pr}_U(\mathbf{s}) \in \mathrm{pr}_U(R^*) = \mathrm{pr}_U(R)$, so there is a tuple $\mathbf{s}_i \in R$ such that $\mathrm{pr}_U(\mathbf{s}) = \mathrm{pr}_U(\mathbf{s}_i)$. This $\mathbf{s}_i$ is an i-approximation of $\mathbf{s}$ in R.

[$\Leftarrow$] Now suppose that $R \in \mathcal{R}$ is an k-ary relation satisfying (1) and (2). Since (by (1)) every k-ary relation in $\mathcal{R}$ properly containing R contains $\mathbf{s}$ and R does not, R is completely $\cap$-irreducible. If R is directly decomposable, then there is a partition $k = U \cup V$ of the index set into two nonempty sets such that, up to a permutation of coordinates, $R = \mathrm{pr}_U(R) \times \mathrm{pr}_V(R)$. Item (2) guarantees the existence of u and v-approximations $\mathbf{s}_u, \mathbf{s}_v \in R$ of $\mathbf{s}$ for any given $u \in U$ and $v \in V$. The fact that they are approximations yields $\mathrm{pr}_U(\mathbf{s}_v) = \mathrm{pr}_U(\mathbf{s})$ and $\mathrm{pr}_V(\mathbf{s}_u) = \mathrm{pr}_V(\mathbf{s})$, so

$$\mathbf{s} = \mathrm{pr}_U(\mathbf{s})\mathrm{pr}_V(\mathbf{s}) = \mathrm{pr}_U(\mathbf{s}_v)\mathrm{pr}_V(\mathbf{s}_u) \in \mathrm{pr}_U(R) \times \mathrm{pr}_V(R) = R,$$

contrary to (1). Hence R is critical. $\square$

Our goal in the remainder of this section is to prove a structure theorem for critical relations in certain algebras. In this paragraph we outline the approach and fix some of the assumptions and notation that will be used. R will always denote a k-ary critical relation of the algebra $\mathbf{A}$, and R will always be assumed to satisfy the $(1, k-1)$-parallelogram property. R^* will denote the unique upper cover of R in the subalgebra lattice of $\mathbf{A}^k$. We shall reduce the representation $\mathbf{R} \leq \mathbf{A}^k$ of R as a subalgebra of $\mathbf{A}^k$ in the following way. First let $A_i := \mathrm{pr}_i(R)$ be the subalgebra of $\mathbf{A}$ that is the projection of R onto its i-th coordinate. This makes $\mathbf{R} \leq_{\mathrm{sd}} \prod_{i<k} \mathbf{A}_i$ a subdirect representation. Next, some general terminology: given a subalgebra $\mathbf{S}$ of an algebra $\mathbf{B}$ and a congruence $\psi \in \mathrm{Con}\,(\mathbf{B})$, let S^ψ denote the union $\bigcup_{s \in S} s/\psi$ of all ψ-classes

that have nonempty intersection with S. Call S *saturated* with respect to ψ (or ψ-saturated) if $S^\psi = S$. Returning to our reduction, let $\Theta = \prod_{i<k} \theta_i \in \mathrm{Con}\left(\prod_{i<k} \mathbf{A}_i\right)$ be the largest product congruence for which R is Θ-saturated. Define $\overline{\mathbf{A}}_i = \mathbf{A}_i/\theta_i$ and let $\nu\colon \prod_{i<k} \mathbf{A}_i \to \prod_{i<k} \overline{\mathbf{A}}_i$ be the natural map between products induced by the quotient maps in each factor. Define $\overline{R} := \nu(R)$ to be the image of R under this map (similarly define $\overline{R^*} := \nu(R^*)$). Then $\overline{\mathbf{R}} \leq_{\mathrm{sd}} \prod_{i<k} \overline{\mathbf{A}_i}$ is a subdirect product representation, and $R = \nu^{-1}(\overline{R})$ is the full inverse image of $\overline{R}$ under ν (since R is Θ-saturated and $\Theta = \ker(\nu)$). Our 'structure theorem' for R really will be a structure theorem for its reduction, $\overline{R}$.

The reduction from R to $\overline{R}$ can be performed on any compatible relation of any algebra, but when $\mathbf{A}$ belongs to a congruence modular variety and R is a critical k-ary relation satisfying the $(1, k-1)$-parallelogram property we are able to show:

- If $k > 1$, then the $\overline{\mathbf{A}}_i$'s are subdirectly irreducible algebras.
- If $k > 1$, then $\overline{R}$ is the graph of a *joint similarity* between the $\overline{\mathbf{A}}_i$'s.
- If $k > 2$, then the $\overline{\mathbf{A}}_i$'s have abelian monoliths.

Before proving our first result we introduce even more notation and terminology. If $\mathbf{S} \leq \prod_I \mathbf{B}_i$ is a subalgebra of a product over the index set I, and $J \subseteq I$, then the kernel of the projection, pr_J, of $\mathbf{S}$ onto the coordinates in J will be denoted η_J (or η_j if $J = \{j\}$). This means that if $\mathbf{s} = (s_i)_{i\in I}, \mathbf{t} = (t_i)_{i\in I} \in S$, then $(\mathbf{s}, \mathbf{t}) \in \eta_J$ if and only if $s_j = t_j$ for all $j \in J$. The projection onto the complementary set of coordinates, $I - J$, will be denoted η'_J (or η'_j). The *i-th coordinate kernel* of $\mathbf{S} \leq \prod_{i<k} \mathbf{B}_i$ is the image under $\mathrm{pr}_i\colon \mathbf{S} \to \mathbf{B}_i$ of the congruence $\eta'_i \in \mathrm{Con}(\mathbf{S})$. That is, the i-th coordinate kernel of $\mathbf{S} \leq \prod_{i<k} \mathbf{B}_i$ is the set of all $(a, b) \in B_i^2$ such that there exist $u_j \in \mathbf{B}_j$ for which $(u_0, \ldots, u_{i-1}, a, u_{i+1}, \ldots, u_{k-1}) \in S$ and $(u_0, \ldots, u_{i-1}, b, u_{i+1}, \ldots, u_{k-1}) \in S$.

Lemma 2.3. *Let R be a k-ary critical relation of $\mathbf{A}$ that satisfies the $(1, k-1)$-parallelogram property. The i-th coordinate kernel of $\mathbf{R} \leq \prod_{i<k} \mathbf{A}_i$, which will be denoted θ_i, is a congruence on $\mathbf{A}_i$. The product $\Theta := \prod_{i<k} \theta_i$ is the largest product congruence on $\prod_{i<k} \mathbf{A}_i$ that saturates R.*

Proof. We first prove that θ_i is a congruence when $i = 0$. The same argument works for the other coordinates.

It follows from the fact that θ_0 is the image of the congruence η'_0 under the surjective homomorphism pr_0 that θ_0 is a reflexive, symmetric, compatible relation on $\mathbf{A}_0$. To see that it is also transitive, choose $(a, b), (b, c) \in \theta_0$. Abbreviating $(a, u_1, \ldots, u_{k-1})$ by $a\mathbf{u}$, this means that there exist $(k-1)$-tuples $\mathbf{u}, \mathbf{v}$ such that $a\mathbf{u}, b\mathbf{u} \in R$ and $b\mathbf{v}, c\mathbf{v} \in R$. Since $a\mathbf{u}, b\mathbf{u}$, and $b\mathbf{v}$ are three vertices of a $(1, k-1)$-parallelogram and lie in R, we must have the fourth vertex $a\mathbf{v}$ in R. Since $a\mathbf{v}, c\mathbf{v} \in R$ we get $(a, c) \in \theta_0$.

Next we prove that R is Θ-saturated. For this we must show that if $\mathbf{a} := (a_0, \ldots, a_{k-1}) \in R$ is Θ-related to $\mathbf{b} := (b_0, \ldots, b_{k-1}) \in \prod_{i<k} A_i$, then $\mathbf{b} \in R$. We

prove by induction that $\mathbf{b}_i := (b_0, \ldots, b_{i-1}, a_i, \ldots, a_{k-1})$ belongs to R for all i satisfying $0 \leq i \leq k$. When $i = 0$ this is the assertion $\mathbf{a} \in R$, which we are assuming, and when $i = k$ this is the assertion $\mathbf{b} \in R$. To prove that $(\mathbf{b}_i \in R) \Rightarrow (\mathbf{b}_{i+1} \in R)$, permute coordinates to put the i-th coordinate first and write $\mathbf{v}$ for the tuple $(b_0, \ldots, b_{i-1}, a_{i+1}, \ldots, a_{k-1})$ of coordinates common to $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$. Thus, we must show that $(a_i\mathbf{v} \in R) \Rightarrow (b_i\mathbf{v} \in R)$ given the fact that $(a_i, b_i) \in \theta_i$. Since $(a_i, b_i) \in \theta_i$ there is a $(k-1)$-tuple $\mathbf{u}$ such that $a_i\mathbf{u}, b_i\mathbf{u} \in R$. Since $\{a_i\mathbf{u}, a_i\mathbf{v}, b_i\mathbf{u}, b_i\mathbf{v}\}$ is a $(1, k-1)$-parallelogram and $a_i\mathbf{u}, a_i\mathbf{v}, b_i\mathbf{u} \in R$ we must have $\mathbf{b}_{i+1} = b_i\mathbf{v} \in R$, which was what was to be proved.

Finally we show that Θ contains every product congruence that saturates R. Let $\Psi = \prod_{i<k} \psi_i \in \mathrm{Con}\left(\prod_{i<k} \mathbf{A}_i\right)$ be any product congruence that saturates R. Choose $(a, b) \in \psi_0$. Since $\mathrm{pr}_0(R) = A_0$ and $a \in A_0$, there is a tuple $a\mathbf{u} \in R$. Since $(a\mathbf{u}, b\mathbf{u}) \in \prod_{i<k} \psi_i = \Psi$ and R is Ψ-saturated we have $b\mathbf{u} \in R$. But $a\mathbf{u}, b\mathbf{u} \in R$ forces $(a, b) \in \theta_0$. Thus $(a, b) \in \psi_0$ implies $(a, b) \in \theta_0$ for any (a, b), proving that $\psi_0 \leq \theta_0$. The same argument works in any other coordinate, so $\Psi \leq \Theta$. □

Lemma 2.4. *Let R be a k-ary critical relation of $\mathbf{A}$ that satisfies the $(1, k-1)$-parallelogram property. If $k > 1$, then each $\overline{\mathbf{A}}_i := \mathbf{A}_i/\theta_i$ is subdirectly irreducible.*

Proof. Since R is a critical relation of $\mathbf{A}$ of arity $k > 1$, there is a smallest k-ary compatible relation R^* properly containing R. Since $R \subseteq \prod_{i<k} A_i$, R is a directly indecomposable relation, and $\prod_{i<k} A_i$ is a directly *de*composable relation when $k > 1$, we have $R \subsetneq \prod_{i<k} A_i$, and consequently $R \subsetneq R^* \subseteq \prod_{i<k} A_i$.

Choose $\mathbf{s} = (s_0, s_1, \ldots, s_{k-1}) \in R^* - R$ and let $\mathbf{s}_0 = (t_0, s_1, \ldots, s_{k-1}) \in R$ be a 0-approximation to $\mathbf{s}$ in R. Both $\mathbf{s}$ and $\mathbf{s}_0$ belong to R^*, hence to $\prod_{i<k} A_i$, according to the conclusion of the previous paragraph. Since $\mathbf{s}_0 \in R$, $\mathbf{s} \notin R$, and R is Θ-saturated for $\Theta = \prod_{i<k} \theta_i$, we have $(\mathbf{s}, \mathbf{s}_0) \notin \prod_{i<k} \theta_i$. But $\mathbf{s}$ and $\mathbf{s}_0$ agree in all coordinates except the 0-th, so $(s_0, t_0) \notin \theta_0$.

Since $(s_0, t_0) \notin \theta_0$, there exist congruences on $\mathbf{A}_0$ that are strictly larger than θ_0. Let ψ be any such congruence, and define $\Psi := \psi \times 0_1 \times \cdots \times 0_{k-1}$. Since $\Psi \not\leq \Theta$, it follows from Lemma 2.3 that the Ψ-saturation R^Ψ of R is strictly larger than R, hence contains R^*, hence contains $\mathbf{s}$. This implies that there is a tuple $\mathbf{u} = (u_0, \ldots, u_{k-1}) \in R$ such that $(\mathbf{s}, \mathbf{u}) \in \Psi$, or equivalently $(s_0, u_0) \in \psi$ and $s_i = u_i$ for $i > 0$. Now $\mathbf{t} = (t_0, s_1, \ldots, s_{k-1}) \in R$ and $\mathbf{u} = (u_0, s_1, \ldots, s_{k-1}) \in R$, so $(t_0, u_0) \in \theta_0$ by the definition of coordinate kernels. From $(s_0, u_0) \in \psi$, $(u_0, t_0) \in \theta_0$, and $\theta_0 \leq \psi$ we derive that $(s_0, t_0) \in \psi$. Combining this with the conclusion of the previous paragraph we have that the pair (s_0, t_0) is not in θ_0 but is in any congruence ψ that is strictly larger than θ_0. It follows that θ_0 is a completely $\cap$-irreducible congruence of $\mathbf{A}_0$ with unique upper cover $\theta_0 \vee \mathrm{Cg}(s_0, t_0)$, and hence that $\overline{\mathbf{A}}_0 = \mathbf{A}_0/\theta_0$ is SI with least nonzero congruence $(\theta_0 \vee \mathrm{Cg}(s_0, t_0))/\theta_0$. The same argument works in any other coordinate. □

Now we are ready to prove the structure theorem. Additional notation used here includes that μ_i is the unique upper cover of θ_i in $\mathrm{Con}\,(\mathbf{A}_i)$ (recall that we proved in Lemma 2.4 that this upper cover is $\theta_i \vee \mathrm{Cg}(s_i, t_i)$ for specified elements s_i, t_i), $\rho_i = (\theta_i : \mu_i)$ is the centralizer of μ_i modulo θ_i, $\overline{\mu}_i = \mu_i/\theta_i$ is the monolith of $\overline{\mathbf{A}}_i$, and $\overline{\rho}_i = \rho_i/\theta_i$ is the centralizer of the monolith.

Theorem 2.5. *Let R be a k-ary critical relation of $\mathbf{A}$ that satisfies the $(1, k-1)$-parallelogram property, and let $\overline{\mathbf{R}} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ be its reduced representation. If $k > 1$ and $\mathbf{A}$ lies in a congruence modular variety, then the following hold.*

(1) *$\overline{\mathbf{R}} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ is a representation of $\overline{\mathbf{R}}$ as a subdirect product of SI algebras.*
(2) *$\overline{\mathbf{R}} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ has trivial coordinate kernels.*
(3) *The projection of $\overline{\mathbf{R}}$ onto any $k-1$ coordinates is 1-1.*
(4) *$\overline{\mathbf{R}}$ is completely $\cap$-irreducible in the subalgebra lattice of $\prod_{i<k} \overline{\mathbf{A}}_i$, and its unique upper cover is $\overline{\mathbf{R}^*}$.*
(5) *If $k > 2$, then the i-th coordinate kernel of $\overline{\mathbf{R}^*} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ is the congruence $\overline{\mu}_i$.*
(6) *$\overline{\mathbf{R}}$ is the graph of a joint similarity between the $\overline{\mathbf{A}}_i$'s.*
(7) *If $k > 2$, then each SI $\overline{\mathbf{A}}_i$ has abelian monolith.*
(8) *The image of the composite map*

$$\overline{R} \hookrightarrow \prod_{i<k} \overline{\mathbf{A}}_i \twoheadrightarrow (\overline{\mathbf{A}}_m/\overline{\rho}_m) \times (\overline{\mathbf{A}}_n/\overline{\rho}_n)$$

is the graph of an isomorphism for any $m, n < k$.

Proof. [(1)] This is a consequence of Lemma 2.4 and the definition of $\overline{R}$.

[(2)] Let $\overline{\theta}_0$ be the 0-th coordinate kernel of $\overline{\mathbf{R}} \leq \prod_{i<k} \overline{\mathbf{A}}_i$. If $(\overline{a}, \overline{b}) \in \overline{\theta}_0$, then there are elements $\overline{u}_i \in A_i$ for $i > 0$ such that $\overline{a}\overline{\mathbf{u}} := (\overline{a}, \overline{u}_1, \ldots, \overline{u}_{k-1}) \in \overline{R}$ and $\overline{b}\overline{\mathbf{u}} := (\overline{b}, \overline{u}_1, \ldots, \overline{u}_{k-1}) \in \overline{R}$. We want to choose preimages in R for the tuples $\overline{a\mathbf{u}}$ and $\overline{b\mathbf{u}}$ with respect to the homomorphism $\nu\colon \prod_{i<k} \mathbf{A}_i \to \prod_{i<k} \overline{\mathbf{A}}_i$. Since R is saturated with respect to $\ker(\nu) = \prod_{i<k} \theta_i$, we can choose preimages $a\mathbf{u}, b\mathbf{u} \in R$ which are equal in all coordinates except the first. Thus a and b are related by the 0-th coordinate kernel of $\mathbf{R} \leq \prod_{i<k} \mathbf{A}_i$, forcing $\overline{a} = \overline{b}$. Since $(\overline{a}, \overline{b}) \in \overline{\theta}_0$ was arbitrary, $\overline{\theta}_0$ is trivial. The same argument works for the other coordinate kernels.

[(3)] This is a restatement of item (2).

[(4)] Every subalgebra $\mathbf{S} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ that properly contains $\overline{R}$ is the image under $\nu\colon \prod_{i<k} \mathbf{A}_i \to \prod_{i<k} \overline{\mathbf{A}}_i$ of a $\ker(\nu)$-saturated subalgebra of $\prod_{i<k} \mathbf{A}_i$ that properly contains R. All of these contain R^*, so their intersection $\mathbf{I}$ also contains R^*. Since $R^* \subseteq I$ and $\mathbf{I}$ is the smallest $\ker(\nu)$-saturated subalgebra of $\prod_{i<k} \mathbf{A}_i$ properly containing R, it follows that $\overline{\mathbf{R}^*} = \nu(\mathbf{R}^*) = \nu(\mathbf{I})$ is the smallest subalgebra of $\prod_{i<k} \overline{\mathbf{A}}_i$ properly containing $\nu(\mathbf{R}) = \overline{\mathbf{R}}$.

[(6)] First we explain what the statement of (6) means. Similarity is an equivalence relation defined on the class of SI algebras in a congruence modular variety. The definition may be found in 10.7 of [5], but we will use the characterization given in Theorem 10.8 of [5] instead: If $\mathbf{B}, \mathbf{C} \in \mathcal{V}$ are SI, a *graph of a similarity from* $\mathbf{B}$ *to* $\mathbf{C}$ is an algebra $\mathbf{G} \in \mathcal{V}$ for which there are congruences $\beta, \gamma, \delta, \varepsilon \in \mathrm{Con}\,(\mathbf{G})$ such that $\mathbf{G}/\beta \cong \mathbf{B}$, $\mathbf{G}/\gamma \cong \mathbf{C}$ and there is a projectivity $\beta^*/\beta \searrow \varepsilon/\delta \nearrow \gamma^*/\gamma$ in $\mathrm{Con}\,(\mathbf{G})$, where β^* and γ^* are the unique upper covers of β and γ respectively. $\mathbf{B}$ and $\mathbf{C}$ are *similar* if there is a graph of a similarity between them.

For a family of SI's, $\{\mathbf{B}_i \mid i \in I\} \subseteq \mathcal{V}_{SI}$, we define a *graph of a joint similarity* between them to be a single algebra $\mathbf{G} \in \mathcal{V}$ for which there are congruences β_i, δ_{ij}, ε_{ij} such that $\mathbf{G}/\beta_i \cong \mathbf{B}_i$ and $\beta_i^*/\beta_i \searrow \varepsilon_{ij}/\delta_{ij} \nearrow \beta_j^*/\beta_j$ for all $i \neq j$.

The claim of (6) is that $\mathbf{G} := \overline{\mathbf{R}}$ is a graph of a joint similarity between the SI's $\mathbf{B}_i := \overline{\mathbf{A}}_i$, $i < k$. To define the congruences that witness this, first choose $\mathbf{s} \in R^* - R$, then select i-approximations $\mathbf{s}_i \in R$ for all i, and finally let $\bar{\mathbf{s}}_i = \nu(\mathbf{s}_i)$ where $\nu\colon \prod_{i<k} \mathbf{A}_i \to \prod_{i<k} \overline{\mathbf{A}}_i$ is the natural map. The congruences that witness joint similarity are $\beta_i := \eta_i$, $\delta_{ij} := 0$ for all $i \neq j$, and $\varepsilon_{ij} := \mathrm{Cg}(\bar{\mathbf{s}}_i, \bar{\mathbf{s}}_j)$.

It suffices to show that $\eta_i^*/\eta_i \searrow \mathrm{Cg}(\bar{\mathbf{s}}_i, \bar{\mathbf{s}}_j)/0$ for any $i \neq j$. We argue only the case where $i = 0, j = 1$, as all cases are the same. Here, if $\mathbf{s} = (s_0, s_1, \ldots, s_{k-1})$, $\mathbf{s}_0 = (t_0, s_1, \ldots, s_{k-1})$ and $\mathbf{s}_1 = (s_0, t_1, \ldots, s_{k-1})$, then we proved in Lemma 2.4 that the unique upper cover of θ_0 is $\mu_0 = \theta_0 \vee \mathrm{Cg}(s_0, t_0)$. Applying ν to these congruences and elements we conclude that the unique upper cover η_0^* $(= \overline{\mu}_0 \times 1 \times \cdots \times 1)$ of the first projection kernel η_0 equals $\eta_0 \vee \mathrm{Cg}(\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_0)$, i.e., $\eta_0^* = \eta_0 \vee \mathrm{Cg}(\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_0)$. So to prove that $\eta_0^*/\eta_0 \searrow \mathrm{Cg}(\bar{\mathbf{s}}_0, \bar{\mathbf{s}}_1)/0$ it remains to show that $\eta_0 \wedge \mathrm{Cg}(\bar{\mathbf{s}}_0, \bar{\mathbf{s}}_1) = 0$. For this, observe that $\mathrm{Cg}(\bar{\mathbf{s}}_0, \bar{\mathbf{s}}_1) \leq \eta'_{\{0,1\}}$, since $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{s}}_1$ agree in all coordinates but the first two, so $\eta_0 \wedge \mathrm{Cg}(\bar{\mathbf{s}}_0, \bar{\mathbf{s}}_1) \leq \eta_0 \wedge \eta'_{\{0,1\}} = \eta'_1$. But η'_1 is the kernel of a projection of $\overline{\mathbf{R}}$ onto $(k-1)$ coordinates, so by item (3) we get that $\eta'_1 = 0$. This completes the proof of (6).

[(7)] Suppose that $\mathbf{G}$ is a graph of a similarity between SI's $\mathbf{B}$ and $\mathbf{C}$ and that $\beta, \gamma, \delta, \varepsilon \in \mathrm{Con}\,(\mathbf{G})$ are such that $\mathbf{G}/\beta \cong \mathbf{B}$, $\mathbf{G}/\gamma \cong \mathbf{C}$, and $\beta^*/\beta \searrow \varepsilon/\delta \nearrow \gamma^*/\gamma$ in $\mathrm{Con}\,(\mathbf{G})$. If the monolith of $\mathbf{B}$ is nonabelian, then so are the perspective quotients β^*/β, ε/δ and γ^*/γ; moreover, by perspectivity, the centralizers $(\beta : \beta^*)$, $(\delta : \varepsilon)$, $(\gamma : \gamma^*)$ are equal. But if β is completely meet irreducible and β^*/β is nonabelian, then $(\beta : \beta^*) = \beta$, and similarly $(\gamma : \gamma^*) = \gamma$. Thus, when one of $\mathbf{B}$ and $\mathbf{C}$ has nonabelian monolith, then both do and $\beta = (\beta : \beta^*) = (\gamma : \gamma^*) = \gamma$.

We showed in (6) that $\overline{R}$ is the graph of a joint similarity between the $\overline{\mathbf{A}}_i$'s, with witnessing congruences $\beta_i := \eta_i$, $\delta_{ij} := 0$ and $\varepsilon_{ij} := \mathrm{Cg}(\bar{\mathbf{s}}_i, \bar{\mathbf{s}}_j)$. If one of the $\overline{\mathbf{A}}_i$'s has nonabelian monolith, then all β_i's ($= \eta_i$'s) must be equal, and must be equal to their intersection, which is zero. Thus, the surjective map $\mathrm{pr}_i\colon \overline{R} \to \overline{\mathbf{A}}_i$ has kernel $\eta_i = 0$ for all i, implying that these maps are isomorphisms. When this happens,

$\overline{R} = \bigcap_{i<j} R_{ij}$ where

$$R_{ij} := \Big\{(a_0, \ldots, a_{i-1}, \mathrm{pr}_i(r), a_{i+1}, \ldots, a_{j-1}, \mathrm{pr}_j(r), a_{j+1}, \ldots, a_{k-1}) \in \prod_{i<k} \overline{A}_i \;\Big|\; r \in \overline{R}\Big\}.$$

But, by (4), $\overline{\mathbf{R}}$ is completely $\cap$-irreducible in the subalgebra lattice of $\prod_{i<k} \overline{\mathbf{A}}_i$. Thus, $\overline{R} = \bigcap_{i<j} R_{ij}$ implies that $\overline{R} = R_{ij}$ for some i and j. But now there can be no coordinates other than i and j, since $\mathrm{pr}_h\colon R_{ij} \to A_h$ is not 1-1 unless $h = i$ or $h = j$ and there are no other coordinates. Thus $k = \{0, \ldots, k-1\} = \{i, j\}$, contrary to our assumption that $k > 2$.

[(5)] We prove that the 0-th coordinate kernel of $\overline{\mathbf{R}^*} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ is $\overline{\mu}_0$. The same argument works for the other coordinates.

The 0-th coordinate kernel consists of all pairs $(\overline{a}, \overline{b}) \in \overline{A}_0 \times \overline{A}_0$ such that there are elements $\overline{u}_i \in A_i$ for $i > 0$ such that $\overline{a}\overline{\mathbf{u}} := (\overline{a}, \overline{u}_1, \ldots, \overline{u}_{k-1}) \in \overline{R^*}$ and $\overline{b}\overline{\mathbf{u}} := (\overline{b}, \overline{u}_1, \ldots, \overline{u}_{k-1}) \in \overline{R^*}$. Choose preimages $a\mathbf{u}, b\mathbf{v} \in R^*$ with respect to the map $\nu\colon \prod_{i<k} \mathbf{A}_i \to \prod_{i<k} \overline{\mathbf{A}}_i$ for these tuples. Since $\mu_0 > \theta_0$, R is not saturated with respect to the product congruence $\Psi := \mu_0 \times 0 \times \cdots \times 0$, so the Ψ-saturation R^Ψ of R contains R^*, hence contains $a\mathbf{u}$ and $b\mathbf{v}$. This means that there are tuples $a'\mathbf{u}, b'\mathbf{v} \in R$ such that $(a, a'), (b, b') \in \mu_0$. Since R is Θ-saturated for $\Theta := \prod_{i<k} \theta_i$, $(u_i, v_i) \in \theta_i$ for all $i > 0$, and $a'\mathbf{u} \in R$, we get that $a'\mathbf{v} \in R$. Now the fact that $a'\mathbf{v}, b'\mathbf{v} \in R$ implies that $(a', b') \in \theta_0 \leq \mu_0$. Therefore $(a, a'), (a', b'), (b', b) \in \mu_0$, showing that $(a, b) \in \mu_0$. Applying ν to this we get that $(\overline{a}, \overline{b}) \in \overline{\mu}_0$. Hence $\overline{\mu}_0$ contains the 0-th coordinate kernel of $\overline{\mathbf{R}^*} \leq \prod_{i<k} \overline{\mathbf{A}}_i$.

The 0-th coordinate kernel of $\overline{\mathbf{R}^*} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ is a tolerance contained in the minimal abelian congruence μ_0 of $\overline{\mathbf{A}}_0$. By modular commutator theory, any tolerance contained in an abelian congruence is a congruence, so the 0-th coordinate kernel is either 0 or it is $\overline{\mu}_0$. The 0-th coordinate kernel cannot be 0, since it contains nontrivial pairs of the form $(s_0/\theta_0, t_0/\theta_0)$ where $\mathbf{s} = (s_0, s_1, \ldots, s_{k-1}) \in R^* - R$ and $(t_0, s_1, \ldots, s_{k-1}) \in R$ is a 0-approximation to $\mathbf{s}$. Thus the 0-th coordinate kernel of $\overline{\mathbf{R}^*} \leq \prod_{i<k} \overline{\mathbf{A}}_i$ is $\overline{\mu}_0$, and (5) is proved.

[(8)] Since $\eta_i^*/\eta_i \searrow \varepsilon_{ij}/0 \nearrow \eta_j^*/\eta_j$ in $\mathrm{Con}\,(\overline{\mathbf{R}})$ for all $i \neq j$, the centralizers $(\eta_i : \eta_i^*)$ are equal for all i. But $(\eta_i : \eta_i^*)$ is the kernel of the composite map

$$\varphi_i\colon \overline{\mathbf{R}} \xrightarrow{\mathrm{pr}_i} \overline{\mathbf{A}}_i \xrightarrow{\mathrm{nat}} \overline{\mathbf{A}}_i/\overline{\rho}_i.$$

The maps φ_m and φ_n are the coordinate maps of the composite map

$$\overline{R} \hookrightarrow \prod_{i<k} \overline{\mathbf{A}}_i \twoheadrightarrow (\overline{\mathbf{A}}_m/\overline{\rho}_m) \times (\overline{\mathbf{A}}_n/\overline{\rho}_n). \tag{2.1}$$

Since φ_m and φ_n are surjective and have the same kernel, the image of the composite map in (2.1) is the graph of an isomorphism. □

3. Parallelogram Terms

The two main results in this section are that k-parallelogram terms enforce the parallelogram property for critical relations (Theorem 3.6) and that a variety has k-parallelogram terms if and only if it has a k-cube term (Theorem 3.5). The first of these is central to this paper, while the second is an important side observation. We prove the second of these first.

Recall that a *k-ary near unanimity term* (or *k-nu term*) for a variety $\mathcal{V}$ is a term U such that:

$$\mathcal{V} \models \mathsf{U}\begin{pmatrix} x & y & y & \cdots & y & y \\ y & x & y & & y & y \\ y & y & x & & y & y \\ \vdots & & & \ddots & & \vdots \\ y & y & y & & x & y \\ y & y & y & \cdots & y & x \end{pmatrix} = \begin{pmatrix} y \\ y \\ y \\ \vdots \\ y \\ y \end{pmatrix}.[1] \tag{3.1}$$

Recall also that a *Maltsev term* for a variety $\mathcal{V}$ is a term M such that:

$$\mathcal{V} \models \mathsf{M}\begin{pmatrix} x & x & y \\ y & x & x \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}. \tag{3.2}$$

Now we describe some terms that generalize both of those in (3.1) and (3.2).

Let $k = \{0, 1, \ldots, k-1\}$. For any subset $U \subseteq k$ let $\chi_U \colon k \to \{0,1\}$ denote its characteristic function. Let $\overline{\chi}_U \colon k \to \{x, y\}$ equal the composite $\beta \circ \chi_U$ where $\beta \colon \{0,1\} \to \{x,y\}$ is the bijection $0 \mapsto y$ and $1 \mapsto x$. (Thus $\overline{\chi}_U$ is also a characteristic function of U, but with values in $\{x,y\}$ instead of $\{0,1\}$.)

Definition 3.1. Let $\mathcal{V}$ be a variety, and let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$ be the free $\mathcal{V}$-algebra over $\{x,y\}$. A *k-cube term* for $\mathcal{V}$ for $k > 1$ is a $\mathcal{V}$-term C for which $\mathsf{C}^{\mathbf{F}^k}(\langle \overline{\chi}_U \rangle_{U \neq \emptyset}) = \overline{\chi}_\emptyset$.

In words, $\mathsf{C}^{\mathbf{F}^k}$ applied to the characteristic functions of nonempty subsets of k produces the characteristic function of the empty set. We do not specify the order of the characteristic functions, so we consider any term obtained from C by permutation of variables to be a k-cube term also.

For example, a 2-cube term is a term C such that

$$\mathsf{C}^{\mathbf{F}^2}\left(\overline{\chi}_{\{0\}}, \overline{\chi}_{\{0,1\}}, \overline{\chi}_{\{1\}}\right) = \overline{\chi}_\emptyset. \tag{3.3}$$

[1]Here and later an equation of this type is to be read along rows, and is intended to be a compact representation of a sequence of identities. In this case, the individual identities satisfied by $\mathcal{V}$ are $\mathsf{U}(x,y,y,\ldots,y,y) = y$, $\mathsf{U}(y,x,y,\ldots,y,y) = y$, $\mathsf{U}(y,y,x,\ldots,y,y) = y$, &c.

If we write characteristic functions as column vectors, $(\overline{\chi}_U(0), \overline{\chi}_U(1), \ldots, \overline{\chi}_U(k-1))^T$, then (3.3) assumes a more familiar form (compare with (3.2)):

$$\mathsf{C}^{\mathbf{F}^2}\begin{pmatrix} x & x & y \\ y & x & x \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}. \tag{3.4}$$

The fact that the rows of (3.4) express equations relating the free generators of $\mathbf{F}_{\mathcal{V}}(x, y)$ means that these equations hold as identities throughout $\mathcal{V}$. Hence, a 2-cube term for $\mathcal{V}$ is nothing other than a Maltsev term. On the other hand, a 3-cube term is something new:

$$\mathsf{C}^{\mathbf{F}^3}\left(\overline{\chi}_{\{0\}}, \overline{\chi}_{\{1\}}, \overline{\chi}_{\{2\}}, \overline{\chi}_{\{0,1\}}, \overline{\chi}_{\{0,2\}}, \overline{\chi}_{\{1,2\}}, \overline{\chi}_{\{0,1,2\}}\right) = \overline{\chi}_{\emptyset},$$

or equivalently

$$\mathcal{V} \models \mathsf{C}\begin{pmatrix} x & y & y & x & x & y & x \\ y & x & y & x & y & x & x \\ y & y & x & y & x & x & x \end{pmatrix} = \begin{pmatrix} y \\ y \\ y \end{pmatrix}. \tag{3.5}$$

If the term C in (3.5) depends only on its first three variables in $\mathcal{V}$, then it is a 3-nu term for $\mathcal{V}$. More generally, a k-nu term is nothing other than a k-cube term that depends only on the variables associated to characteristic functions of the singletons.

Definition 3.2. A *k-edge term* for a variety $\mathcal{V}$ for $k > 1$ is a term E for which

$$\mathsf{E}^{\mathbf{F}^k}\left(\overline{\chi}_{\{0\}}, \overline{\chi}_{\{0,1\}}, \overline{\chi}_{\{1\}}, \overline{\chi}_{\{2\}}, \ldots, \overline{\chi}_{\{k-1\}}\right) = \overline{\chi}_{\emptyset},$$

or equivalently

$$\mathcal{V} \models \mathsf{E}\begin{pmatrix} x & x & y & y & \cdots & y \\ y & x & x & y & & y \\ y & y & y & x & & y \\ \vdots & & & & \ddots & \vdots \\ y & y & y & y & \cdots & x \end{pmatrix} = \begin{pmatrix} y \\ y \\ y \\ y \\ y \end{pmatrix}. \tag{3.6}$$

In words, a k-edge term is a term which when applied to the characteristic functions of all of the singletons, $\{i\}$, and exactly one doubleton (or "edge"), $\{0, 1\}$, produces the characteristic function of the empty set. As such, it is simply a k-cube term that depends only on the variables associated to singleton sets and the doubleton $\{0, 1\}$. Here we *do* specify the order of the variables, because later we will need to refer to the "edge term identities" without ambiguity. The order we have chosen makes it clear that a Maltsev term is nothing other than a 2-edge term, and that a k-nu term is nothing other than a k-edge term that does not depend on its second variable.

Both k-cube terms and k-edge terms were introduced in [1], where the following results were proved.

Theorem 3.3. *Let $\mathcal{V}$ be a variety and $k > 1$ be an integer.*

(1) *$\mathcal{V}$ has a k-cube term if and only if it has a k-edge term.*
(2) *If $\mathcal{V}$ has a k-cube term, then it is congruence modular.*

Definition 3.4. Let m and n be positive integers and let $k = m + n$. An (m, n)-*parallelogram term* (or k-*parallelogram term*) for $\mathcal{V}$ is a term $\mathsf{P} = \mathsf{P}_{m,n}$ for which

$$\mathcal{V} \models \mathsf{P}\left(\begin{array}{ccc|cccccccc} x & x & y & z & y & \cdots & y & y & \cdots & y & y \\ x & x & y & y & z & & y & y & & y & y \\ & \vdots & & \vdots & & \ddots & & & & & \vdots \\ x & x & y & y & y & & z & y & & y & y \\ \hline y & x & x & y & y & & y & z & & y & y \\ & \vdots & & \vdots & & & & & \ddots & & \vdots \\ y & x & x & y & y & & y & y & & z & y \\ y & x & x & y & y & \cdots & y & y & \cdots & y & z \end{array}\right) = \begin{pmatrix} y \\ y \\ \vdots \\ y \\ y \\ \vdots \\ y \\ y \end{pmatrix}. \tag{3.7}$$

Here P is $(k+3)$-ary, the rightmost block of variables is a $k \times k$ array, the upper left block is $m \times 3$ and the lower left block is $n \times 3$.

In this definition, if $m+n = k = m'+n'$ for positive pairs (m, n) and (m', n'), then $\mathsf{P}_{m,n}$ and $\mathsf{P}_{m',n'}$ satisfy different identities. But we will see next that a variety has a term satisfying the identities of $\mathsf{P}_{m,n}$ if and only if it has a term satisfying the identities of $\mathsf{P}_{m',n'}$, which is why we are justified in referring to either as a k-parallelogram term (with no reference to (m, n)).

Theorem 3.5. *The following are equivalent for a variety $\mathcal{V}$ and an integer $k > 1$.*
(i) *$\mathcal{V}$ has (m, n)-parallelogram terms for all pairs (m, n) of positive integers satisfying $m + n \geq k$.*
(ii) *$\mathcal{V}$ has an (m, n)-parallelogram term for some pair (m, n) of positive integers satisfying $m + n = k$.*
(iii) *$\mathcal{V}$ has a k-cube term.*
(iv) *$\mathcal{V}$ has a k-edge term.*

Proof. [(i)⇒(ii)] This is a tautology.

[(ii)⇒(iii)] Let P be an (m, n)-parallelogram term for $\mathcal{V}$. Substituting x for each z in the identities (3.7) shows that

$$\mathsf{P}\left(\overline{\chi}_{\{0,\ldots,m-1\}}, \overline{\chi}_{\{0,\ldots,k-1\}}, \overline{\chi}_{\{m,\ldots,k-1\}}, \overline{\chi}_{\{0\}}, \overline{\chi}_{\{1\}}, \ldots, \overline{\chi}_{\{k-1\}}\right) = \overline{\chi}_{\emptyset}.$$

Since m and n are positive, each input to P is the characteristic function of a nonempty set. Therefore P itself is a k-cube term (depending on at most $k + 3$ of its variables).

[(iii)⇒(iv)] This is from Theorem 3.3 (1).

[(iv)⇒(i)] Assume that $\mathcal{V}$ is any variety with a k-edge term E, and that $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y, z)$ is the free $\mathcal{V}$-algebra over $\{x, y, z\}$. We first show that it is possible to construct a $(1,k-1)$-parallelogram term for $\mathcal{V}$ from E. The $(1,k-1)$-parallelogram identities (3.7) express the fact that the column $(y,y,\dots,y)^T$ lies in the subalgebra of $\mathbf{F}^k$ generated by the columns that appear in the array of inputs to P in (3.7), i.e.,

$$\begin{pmatrix} y\\ y\\ \vdots\\ y\\ y\end{pmatrix} \in \left\langle \begin{pmatrix} x\\ y\\ \vdots\\ y\\ y\end{pmatrix}, \begin{pmatrix} x\\ x\\ \vdots\\ x\\ x\end{pmatrix}, \begin{pmatrix} y\\ x\\ \vdots\\ x\\ x\end{pmatrix}, \begin{pmatrix} z\\ y\\ \vdots\\ y\\ y\end{pmatrix}, \begin{pmatrix} y\\ z\\ \vdots\\ y\\ y\end{pmatrix}, \dots, \begin{pmatrix} y\\ y\\ \vdots\\ z\\ y\end{pmatrix}, \begin{pmatrix} y\\ y\\ \vdots\\ y\\ z\end{pmatrix} \right\rangle. \tag{3.8}$$

To see that (3.8) holds, use the edge identities (3.6) to verify that

$$\mathsf{E}\begin{pmatrix} x & y & y & y & y & \cdots & y\\ y & z & z & y & y & & y\\ y & y & y & z & y & & y\\ y & y & y & y & z & & y\\ \vdots & & & & & \ddots & \vdots\\ y & y & y & y & y & \cdots & z \end{pmatrix} = \begin{pmatrix} a\\ y\\ y\\ y\\ \vdots\\ y\end{pmatrix} \tag{3.9}$$

where $a = \mathsf{E}(x,y,y,y,\dots,y)$, and

$$\mathsf{E}\begin{pmatrix} x & y & y & y & y & \cdots & y\\ x & x & z & z & z & & z\\ x & x & y & y & y & & y\\ x & x & y & y & y & & y\\ \vdots & & & & & \ddots & \vdots\\ x & x & y & y & y & \cdots & y \end{pmatrix} = \begin{pmatrix} a\\ z\\ y\\ y\\ \vdots\\ y\end{pmatrix} \tag{3.10}$$

where again $a = \mathsf{E}(x,y,y,y,\dots,y)$. Finally, substituting the output tuples from (3.9) and (3.10) into the first two positions of E and subalgebra generators from (3.8) into the other positions we obtain

$$\mathsf{E}\begin{pmatrix} a & a & y & y & y & \cdots & y\\ y & z & z & y & y & & y\\ y & y & y & z & y & & y\\ y & y & y & y & z & & y\\ \vdots & & & & & \ddots & \vdots\\ y & y & y & y & y & \cdots & z \end{pmatrix} = \begin{pmatrix} y\\ y\\ y\\ y\\ \vdots\\ y\end{pmatrix}, \tag{3.11}$$

showing that (3.8) holds. Working through the calculation we find that we have shown that if $\mathsf{P}(x,y,z,w_0,\dots,w_{k-1})$ is defined to be the term

$$\mathsf{E}(\mathsf{E}(x,w_1,w_1,w_2,\dots,w_{k-1}),\mathsf{E}(y,z,w_1,w_1,\dots,w_1),w_1,w_2,\dots,w_{k-1}),$$

then P is a $(1,k-1)$-parallelogram term.

Next we prove that if $\mathcal{V}$ is any variety with a (m,n)-parallelogram term P for some positive m and n, then it also has a $(m+1,n-1)$-parallelogram term Q, provided $n-1$ is positive. The identities that we assume to hold for P are:

$$\mathcal{V} \models \mathsf{P}\left(\begin{array}{ccc|ccccccc} x & x & y & z & \cdots & y & y & y & \cdots & y \\ & \vdots & & \vdots & \ddots & & & & & \vdots \\ x & x & y & y & & z & y & y & & y \\ \hline y & x & x & y & & y & z & y & & y \\ \hline y & x & x & y & & y & y & z & & y \\ & \vdots & & \vdots & & & & & \ddots & \vdots \\ y & x & x & y & \cdots & y & y & y & \cdots & z \end{array}\right) = \begin{pmatrix} y \\ \vdots \\ y \\ y \\ y \\ \vdots \\ y \end{pmatrix}, \tag{3.12}$$

where we have divided the array into m rows, followed by one row, followed by $n-1$ rows. We need to construct Q so that it satisfies the following identities:

$$\mathcal{V} \models \mathsf{Q}\left(\begin{array}{ccc|ccccccc} x & x & y & z & \cdots & y & y & y & \cdots & y \\ & \vdots & & \vdots & \ddots & & & & & \vdots \\ x & x & y & y & & z & y & y & & y \\ \hline x & x & y & y & & y & z & y & & y \\ \hline y & x & x & y & & y & y & z & & y \\ & \vdots & & \vdots & & & & & \ddots & \vdots \\ y & x & x & y & \cdots & y & y & y & \cdots & z \end{array}\right) = \begin{pmatrix} y \\ \vdots \\ y \\ y \\ y \\ \vdots \\ y \end{pmatrix}, \tag{3.13}$$

where the only difference from (3.12) occurs at the beginning of the $(m+1)$-rst row.

Much like the earlier part of the proof, showing the existence of some Q satisfying these identities is equivalent to showing that the column $(y,y,\ldots,y)^T$ in $\mathbf{F}^k$ can be generated via P from the $k+3$ columns of the array of inputs to Q in (3.13). To start we have:

$$\mathsf{P}\left(\begin{array}{ccc|ccccccc} x & x & y & z & \cdots & y & y & y & \cdots & y \\ & \vdots & & \vdots & \ddots & & & & & \vdots \\ x & x & y & y & & z & y & y & & y \\ \hline x & x & y & y & & y & z & y & & y \\ \hline y & x & x & y & & y & y & z & & y \\ & \vdots & & \vdots & & & & & \ddots & \vdots \\ y & x & x & y & \cdots & y & y & y & \cdots & z \end{array}\right) = \begin{pmatrix} y \\ \vdots \\ y \\ a \\ y \\ \vdots \\ y \end{pmatrix}, \tag{3.14}$$

where $a = \mathsf{P}(x,x,y,y,\ldots,z,\ldots,y)$ with the lone z in the $(m+4)$-th position. Then we have

$$
\tag{3.15}
\mathsf{P}\left(\begin{array}{ccc|ccccccc}
x & x & y & y & \cdots & y & y & y & \cdots & y \\
 & \vdots & & \vdots & \ddots & & & & & \vdots \\
x & x & y & y & & y & y & y & & y \\
\hline
x & x & y & y & & y & z & y & & y \\
\hline
x & x & x & x & & x & y & x & & x \\
 & \vdots & & \vdots & & & & & \ddots & \vdots \\
x & x & x & x & \cdots & x & y & x & \cdots & x
\end{array}\right) = \begin{pmatrix} y \\ \vdots \\ y \\ a \\ x \\ \vdots \\ x \end{pmatrix},
$$

where $a = \mathsf{P}(x, x, y, y, \ldots, z, \ldots, y)$ is the same as before. In (3.15), all entries in the last $n - 1$ rows are x except those in the $(m + 4)$-th column, which are y. All entries in higher rows are the same as those in (3.14) except those in the upper right $m \times (m + n)$ block, which are all y. The identities used to establish (3.15) are some consequences of the (m, n)-parallelogram identities obtained by identifying variables.

Using the output tuples from (3.14) and (3.15) together with other columns from the array of inputs from (3.13) we find that

$$
\tag{3.16}
\mathsf{P}\left(\begin{array}{ccc|cccccccc}
y & y & y & y & y & \cdots & z & y & \cdots & y \\
 & \vdots & & \vdots & & \iddots & & & & \vdots \\
y & y & y & y & z & & y & y & & y \\
\hline
a & a & y & z & y & & y & y & & y \\
\hline
y & x & x & y & y & & y & z & & y \\
 & \vdots & & \vdots & & & & & \ddots & \vdots \\
y & x & x & y & y & \cdots & y & y & \cdots & z
\end{array}\right) = \begin{pmatrix} y \\ \vdots \\ y \\ y \\ y \\ \vdots \\ y \end{pmatrix}.
$$

Working through the calculation, we have shown that if $\mathsf{Q}(x, y, z, w_0, \ldots, w_{k-1})$ is defined to be the term

$$
\mathsf{P}(\mathsf{P}(x, y, z, w_0, \ldots, w_{k-1}), \mathsf{P}(y, y, z, z, \ldots, w_m, \ldots, z), z, w_m, \ldots, w_0, w_{m+1}, \ldots, w_{k-1}),
$$

then Q is an $(m + 1, n - 1)$-parallelogram term.

Our arguments show that if $\mathcal{V}$ has a k-edge term, then it has (m, n)-parallelogram terms for all positive pairs (m, n) such that $m + n = k$. But any k-edge term is also a k'-edge term that does not depend on its last $k' - k$ variables whenever $k' \geq k$. Thus, if $\mathcal{V}$ has a k-edge term for some $k > 1$, then it has (m, n)-parallelogram terms for all positive pairs (m, n) for which $m + n = k' \geq k$. □

The next theorem indicates one way in which k-parallelogram terms generalize Maltsev terms and k-nu terms.

Theorem 3.6. *Let $\mathcal{V}$ be a variety.*

(1) *$\mathcal{V}$ has a k-nu term if and only if no algebra $\mathbf{A} \in \mathcal{V}$ has a critical relation of arity at least k.*

(2) *$\mathcal{V}$ has a Maltsev term if and only if for every $\mathbf{A} \in \mathcal{V}$ it is the case that every critical relation of $\mathbf{A}$ satisfies the parallelogram property.*
(3) *$\mathcal{V}$ has a k-parallelogram term if and only if for every $\mathbf{A} \in \mathcal{V}$ it is the case that every critical relation of $\mathbf{A}$ of arity at least k satisfies the parallelogram property.*

Proof. [(1), $\Rightarrow$] Let U be a k-nu term for $\mathcal{V}$. Choose any algebra $\mathbf{A} \in \mathcal{V}$, any compatible relation R of $\mathbf{A}$ of arity $\ell \geq k$, and any tuple $\mathbf{a} \in A^\ell$. Suppose that R contains i-approximations $\mathbf{a}_i$ for all $i = 0, \ldots, k-1$. Then since U is a k-nu term and R is compatible with U we have $\mathbf{a} = \mathsf{U}(\mathbf{a}_0, \ldots, \mathbf{a}_{k-1}) \in R$. In words, if R contains i-approximations to $\mathbf{a}$ for at least k distinct i's, then R must contain $\mathbf{a}$. It follows from Lemma 2.1 that R is not critical. Since $\mathbf{A}$ and R were arbitrary, $\mathcal{V}$ has no algebra with a critical relation of arity at least k.

[(1), $\Leftarrow$] We prove the contrapositive. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. If $\mathcal{V}$ does not have a k-nu term, then the subalgebra $\mathbf{S} \leq \mathbf{F}^k$ that is generated by the characteristic functions of the singletons, $\overline{\chi}_{\{i\}}$, does not contain the characteristic function of the empty set, $\overline{\chi}_{\emptyset}$. Extend $\mathbf{S}$ to a subalgebra $\mathbf{R} \leq \mathbf{F}^k$ that is maximal for not containing $\overline{\chi}_{\emptyset}$. Then R is a compatible k-ary relation on $\mathbf{F} \in \mathcal{V}$. Since R contains the i-approximations $\overline{\chi}_{\{i\}}$ to $\overline{\chi}_{\emptyset}$ and is maximal as a k-ary compatible relation on $\mathbf{F}$ for the property $\overline{\chi}_{\emptyset} \notin R$, it follows from Lemma 2.1 that R is a critical k-ary relation.

[(2), $\Rightarrow$] Suppose that M is a Maltsev term for $\mathcal{V}$, that $\mathbf{A} \in \mathcal{V}$, and that R is any compatible relation of $\mathbf{A}$. Permuting coordinates if necessary, a typical parallelogram has vertices $\mathbf{ac}, \mathbf{ad}, \mathbf{bc}$, and $\mathbf{bd}$. If the first three are in R, then the fourth is also in R since:

$$R \ni \mathsf{M}(\mathbf{ad}, \mathbf{ac}, \mathbf{bc}) = \mathsf{M}(\mathbf{a}, \mathbf{a}, \mathbf{b})\mathsf{M}(\mathbf{d}, \mathbf{c}, \mathbf{c}) = \mathbf{bd}.$$

[(2), $\Leftarrow$] As in (1), we prove the contrapositive. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x, y)$. If $\mathcal{V}$ does not have a Maltsev term, then the subalgebra $\mathbf{S} \leq \mathbf{F}^2$ that is generated by xy, xx and yx does not contain yy. Extend $\mathbf{S}$ to a subalgebra $\mathbf{R} \leq \mathbf{F}^2$ that is maximal for $yy \notin R$. Since R contains the 0 and 1-approximations xy and yx to yy, it follows from from Lemma 2.1 that R is a critical binary relation. This relation fails the parallelogram property, since it contains only the first three vertices of the parallelogram $\{xx, xy, yx, yy\}$.

[(3), $\Rightarrow$] Suppose that $\mathcal{V}$ has a k-parallelogram term, that $\mathbf{A} \in \mathcal{V}$, and that R is any critical relation of $\mathbf{A}$ of arity $\ell \geq k$. Permuting coordinates if necessary, a typical parallelogram in $\mathbf{A}^\ell$ has vertices $\mathbf{ac}, \mathbf{ad}, \mathbf{bc}$, and $\mathbf{bd}$ where $|\mathbf{a}| = m = |\mathbf{b}|$, $|\mathbf{c}| = n = |\mathbf{d}|$ and m and n are positive integers satisfying $m + n = \ell$. Assume, for the sake of obtaining a contradiction, that $\Pi := \{\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{bd}\}$ is a parallelogram with $\mathbf{ac}, \mathbf{ad}, \mathbf{bc} \in R$ while $\mathbf{bd} \notin R$.

Since R is completely $\cap$-irreducible, it has a unique upper cover R^* in the $\cap$-semilattice of ℓ-ary compatible relations of $\mathbf{A}$. Our first objective will be to deduce from the existence of Π that there is a parallelogram Π' with three vertices in R and the fourth in $R^* - R$. Choose any $\mathbf{s} \in R^* - R$. Since $\mathbf{bd} \notin R$, the subalgebra of $\mathbf{A}^\ell$ generated by $R \cup \{\mathbf{bd}\}$ contains the upper cover R^*, hence contains the tuple $\mathbf{s}$. Therefore there exist tuples $\mathbf{e}_0, \ldots, \mathbf{e}_{h-1} \in R$ and an $(h+1)$-ary term t such that $\mathbf{s} = t(\mathbf{bd}, \mathbf{e}_0, \ldots, \mathbf{e}_{h-1})$ in $\mathbf{A}^\ell$. If we apply the polynomial $f(x) := t(x, \mathbf{e}_0, \ldots, \mathbf{e}_{h-1})$ to the four vertices of the rectangle

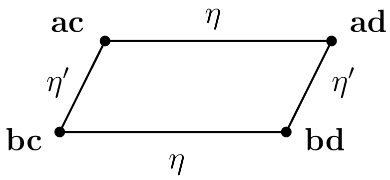

FIGURE 5. An (m,n)-parallelogram

we get

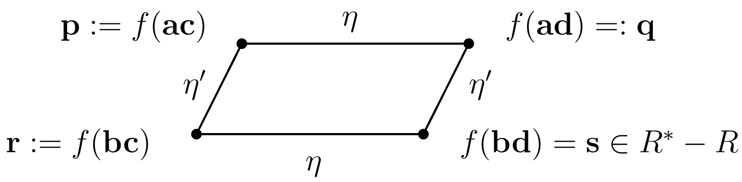

FIGURE 6. Another (m,n)-parallelogram

Because of the way $\mathbf{p}, \mathbf{q}, \mathbf{r}$ and $\mathbf{s}$ are related by η and η', the set $\Pi' := \{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}\}$ is also an (m,n)-parallelogram. Because of the way $\mathbf{p}, \mathbf{q}$, and $\mathbf{r}$ were generated via t from $\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{e}_i \in R$ we have $\mathbf{p}, \mathbf{q}, \mathbf{r} \in R$, while we have arranged that $\mathbf{s} \in R^* - R$. Thus, we have achieved our first objective.

From Lemma 2.1, the fact that $\mathbf{s} \in R^* - R$ implies that R contains i-approximations to $\mathbf{s}$ for all i; choose some and denote them by $\mathbf{s}_0, \ldots, \mathbf{s}_{\ell-1}$. Now, if $\mathsf{P} = \mathsf{P}_{m,n}$ is an (m,n)-parallelogram term, then a direct application of the parallelogram identities shows that

$$\mathsf{P}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}_0, \ldots, \mathbf{s}_{\ell-1}) = \mathbf{s}.$$

Since the inputs to P are in R and the output is not, this contradicts the compatibility of R with P.

[(3), $\Leftarrow$] Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y,z)$. Let $x\mathbf{x}, x\mathbf{y}, y\mathbf{x}$ and $y\mathbf{y}$ denote the tuples in F^k whose 0-th coordinates are x, x, y and y respectively, and whose i-th coordinates are x, y, x and y respectively for all larger i. Let $\mathbf{y}_i$ denote the tuple in F^k whose i-th coordinate is z and whose other coordinates are y. If $\mathcal{V}$ does not have a $(1, k-1)$-parallelogram term, then the subalgebra $\mathbf{S} \leq \mathbf{F}^k$ that is generated by $\{x\mathbf{y}, x\mathbf{x}, y\mathbf{x}, \mathbf{y}_0, \ldots, \mathbf{y}_{k-1}\}$ does not contain $y\mathbf{y}$. Extend $\mathbf{S}$ to a subalgebra $\mathbf{R} \leq \mathbf{F}^k$ that is maximal for $y\mathbf{y} \notin R$. Since R contains the i-approximations $\mathbf{y}_i$ to $y\mathbf{y}$ it follows from from Lemma 2.1 that R is a critical relation. R has arity k and it fails the parallelogram property, since it contains only the first three vertices of the parallelogram $\{x\mathbf{x}, x\mathbf{y}, y\mathbf{x}, y\mathbf{y}\}$. This completes the proof of (3). □

We have just proved that, when one quantifies over all algebras in a variety, certain properties of clones (e.g., the existence of a parallelogram term) are equivalent to certain properties of local clones (e.g., critical relations of sufficiently large arity satisfy the parallelogram property). One cannot expect such a connection between clones and local clones of single algebras. In fact, all parts of the previous theorem are false for single algebras. For example, if $\mathbf{A}$ is an infinite algebra with universe A whose basic operations are all operations on A with finite range, then $\mathbf{A}$ is locally primal. Hence the only critical relations of $\mathbf{A}$ are the unary relation A and the binary equality relation. These relations are bounded in arity and satisfy the parallelogram property, yet $\mathbf{A}$ has no nu term, no Maltsev term, and no parallelogram term, since the only term operations of $\mathbf{A}$ with infinite range are the projection operations. But if $\mathbf{A}$ is a finite algebra, then there is something that can be said.

Theorem 3.7. *Let* $\mathbf{A}$ *be a finite algebra.*

(1) $\mathbf{A}$ *has a k-nu term for some k if and only if there is a bound on the arity of the critical relations of* $\mathbf{A}$.
(2) $\mathbf{A}$ *has a Maltsev term if and only if every critical relation of* $\mathbf{A}$ *satisfies the parallelogram property.*
(3) $\mathbf{A}$ *has a k-parallelogram term for some k if and only if every critical relation of* $\mathbf{A}$ *of sufficiently large arity satisfies the parallelogram property.*

Proof. The forward direction of each part of this theorem follows from the forward direction of the corresponding part of Theorem 3.6. The backward directions also follow from the proof of Theorem 3.6 with the addition of one new idea. It will be enough to illustrate the argument for item (3) only.

[(3), $\Leftarrow$] We prove the contrapositive, so assume that $\mathbf{A}$ has no parallelogram term. Let $\mathbf{F} \leq \mathbf{A}^m$ be a 3-generated free algebra in the variety generated by $\mathbf{A}$. Then $\mathbf{F}$ also has no parallelogram term, so the proof of Theorem 3.6 shows that for any k the subalgebra $\mathbf{S}$ of $\mathbf{F}^k$ generated by $\{x\mathbf{y}, x\mathbf{x}, y\mathbf{x}, \mathbf{y}_0, \ldots, \mathbf{y}_{k-1}\}$ does not contain $y\mathbf{y}$. Here we will represent elements of $F \subseteq A^m$ as column vectors of length m with entries from A, and will represent k-tuples $a\mathbf{b} \in F^k$ as the corresponding $m \times k$ matrices of elements of A. The columns associated to the free generators $x, y, z \in F$ will be $(a_0, \ldots, a_{m-1})^T$, $(b_0, \ldots, b_{m-1})^T$, and $(c_0, \ldots, c_{m-1})^T$ respectively, so the matrices associated with generators $x\mathbf{y}$ and $\mathbf{y}_0$ of S are

$$x\mathbf{y} = \begin{bmatrix} a_0 & b_0 & & b_0 \\ \vdots & \vdots & \dots & \vdots \\ a_{m-1} & b_{m-1} & & b_{m-1} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_0 = \begin{bmatrix} c_0 & b_0 & & b_0 \\ \vdots & \vdots & \dots & \vdots \\ c_{m-1} & b_{m-1} & & b_{m-1} \end{bmatrix},$$

while the matrices for the other generators follow the same pattern.

In the proof of Theorem 3.6 we extended $\mathbf{S}$ to a subalgebra $\mathbf{R} \leq \mathbf{F}^k$ maximal for the property $y\mathbf{y} \notin R$, and found that R was a k-ary critical relation of $\mathbf{F}$ that does not satisfy the parallelogram property. We cannot do that here, since there is no guarantee that such a relation will be a critical relation of $\mathbf{A}$. What fails to be true now is that, although $\mathbf{S}$ contains i-approximations to $y\mathbf{y}$ for all i when considered as a subalgebra of $\mathbf{F}^k$, when considered as a subalgebra of $\mathbf{A}^{m\times k}$ it need not contain "(i,j)-approximations" to the matrix

$$y\mathbf{y} = \begin{bmatrix} b_0 & b_0 & & b_0 \\ \vdots & \vdots & \dots & \vdots \\ b_{m-1} & b_{m-1} & & b_{m-1} \end{bmatrix} \tag{3.17}$$

for all i, j. (An (i,j)-*approximation* of $y\mathbf{y}$ is a matrix in $A^{m\times k}$ that agrees with the matrix in (3.17) in all coordinates except the (i,j)-th.)

So choose a subset of coordinates $U \subseteq m \times k = \{0,\dots,m-1\} \times \{0,\dots,k-1\}$ satisfying the following properties:

(i) U contains every pair $(0,j)$, $j < k$,
(ii) $\mathrm{pr}_U(y\mathbf{y}) \notin \mathrm{pr}_U(S)$, and
(iii) If $V \subsetneq U$, then $\mathrm{pr}_V(y\mathbf{y}) \in \mathrm{pr}_V(S)$.

To see that such a set U exists, notice that $U := m \times k$ satisfies items (i) and (ii), since $y\mathbf{y} \notin S$. Any set minimal with respect to satisfying (i) and (ii) must also satisfy (iii) except possibly the set $U = \{0\} \times k$. But this set does in fact satisfy (iii), since $\mathrm{pr}_V(y\mathbf{y}) = \mathrm{pr}_V(\mathbf{y}_i) \in \mathrm{pr}_V(S)$ if $V \subseteq (\{0\} \times k) - \{(0,i)\}$. Thus, U can be chosen to be any set minimal with respect to satisfying (i) and (ii).

If $S' := \mathrm{pr}_U(S)$ where U satisfies (i), (ii) and (iii) from the previous paragraph, then S' contains three vertices $\mathrm{pr}_U(x\mathbf{y}), \mathrm{pr}_U(x\mathbf{x})$, and $\mathrm{pr}_U(y\mathbf{x})$ of a parallelogram, but not the fourth, $\mathrm{pr}_U(y\mathbf{y})$, according to property (ii). By property (iii), S' contains (i,j)-approximations to $\mathrm{pr}_U(y\mathbf{y})$ for all $(i,j) \in U$. Therefore, if $\mathbf{R} \leq \mathbf{A}^U$ is an extension of $\mathbf{S}'$ that is maximal with respect to not containing $\mathrm{pr}_U(y\mathbf{y})$, then R is a critical relation of $\mathbf{A}$ and does not satisfy the parallelogram property. By property (i) above, the arity of R is somewhere in the interval $[k, mk]$. Thus, we have shown that if $\mathbf{A}$ does not have a k-parallelogram term and the 3-generated free algebra in $\mathcal{V}(\mathbf{A})$ is embeddable in $\mathbf{A}^m$, then $\mathbf{A}$ has a critical relation that does not satisfy the parallelogram property whose arity is in the interval $[k, mk]$. If $\mathbf{A}$ has no parallelogram term at all, then it has critical relations of arbitrarily large arity that fail the parallelogram property. $\square$

4. Applications

The following result is an immediate consequence of Theorems 3.3 (2), 3.5, 3.6 (3) and Theorem 2.5.

Theorem 4.1. *Let $\mathcal{V}$ be a variety with a k-parallelogram term. If $\mathbf{A} \in \mathcal{V}$ and R is a critical relation of $\mathbf{A}$ of arity $\ell \geq \max(k, 3)$, then the reduction $\overline{R}$ has the structure described in Theorem 2.5.*

(Only items (5) and (7) of Theorem 2.5 require the arity of R to be at least 3.)

We wish to apply Theorem 4.1 to derive results about the clones of algebras with parallelogram terms. The first important result is the following, which is also proved in [1, 8] using k-edge terms in place of k-parallelogram terms.

Corollary 4.2. *If $\mathcal{V}$ is a congruence distributive variety with a k-parallelogram term, then $\mathcal{V}$ has a $\max(k, 3)$-nu term.*

Proof. According to Theorem 4.1, any critical relation of arity at least $\max(k, 3)$ on any $\mathbf{A} \in \mathcal{V}$ involves SI sections of $\mathbf{A}$ with abelian monoliths. Since no SI in a congruence distributive variety has an abelian monolith, it must be that no $\mathbf{A} \in \mathcal{V}$ has a critical relation of arity at least $\max(k, 3)$. By Theorem 3.6 (1), $\mathcal{V}$ has a $\max(k, 3)$-nu term. □

The rest of this paper will be devoted to proving the following result.

Theorem 4.3. *For a fixed integer $k > 1$ and fixed finite set A there are only finitely many clones of algebras on A that have a k-parallelogram term and generate a residually small variety.*

It was mentioned in the introduction that on a finite set A a set $\mathcal{C}$ of operations is a clone if and only if $\mathcal{C}$ is Galois closed, meaning that $\mathcal{C}^{\perp\perp} = \mathcal{C}$. Moreover, each clone $\mathcal{C}$ corresponds to a unique Galois closed set of relations, or relational clone, namely $\mathcal{R} := \mathcal{C}^{\perp}$, and $\mathcal{R}$ determines $\mathcal{C}$ by $\mathcal{C} = \mathcal{R}^{\perp}$. The following theorem is well known, see [2, 6, 11].

Theorem 4.4. *Let $\mathcal{C}$ be a clone and let $\mathcal{R} := \mathcal{C}^{\perp}$ be the corresponding relational clone on a finite set A. The following conditions on a subset $\mathcal{S}$ of $\mathcal{R}$ are equivalent:*

(i) $\mathcal{S}^{\perp} = \mathcal{C}$*;*

(ii) $\mathcal{S}^{\perp\perp} = \mathcal{R}$*;*

(iii) *every relation in $\mathcal{R}$ can be obtained from relations in $\mathcal{S}$ and from the equality relation on A by the relational clone operations of finite direct product, intersection of relations of the same arity, permutation of coordinates, and projection onto a subset of coordinates;*

(iv) *every relation in $\mathcal{R}$ is definable in the relational structure $(A; \mathcal{S})$ by a positive primitive formula.*

We also argued in the introduction that every member of $\mathcal{R}$ can be obtained from critical relations in $\mathcal{R}$ via the operations of finite direct product and intersection of relations of the same arity. Hence we get the following.

Corollary 4.5. *(i)–(iv) in Theorem 4.4 are also equivalent to the condition that*

(v) *every critical relation in $\mathcal{R}$ is definable in the relational structure $(A;\mathcal{S})$ by a positive primitive formula.*

To prove Theorem 4.3 we want to use criterion (v) and our understanding of the structure of critical relations of algebras with k-parallelogram terms (Theorem 4.1). However, our structure theorem for critical relations (Theorem 2.5) describes the structure of the reduction $\overline{R}$ of a critical relation R rather than R itself. Therefore before proving Theorem 4.3 we want to adapt the use of positive primitive definitions to this situation.

For a cardinal λ a *multisorted relational signature* $\mathbb{S}$ *with* λ *sorts* is a set of (finitary) relation symbols such that to each relation symbol U in $\mathbb{S}$ there is an associated arity n_{U} (≥ 1) and a function $\sigma_{\mathsf{U}}\colon n_{\mathsf{U}} \to \lambda$ specifying for each $i < n_{\mathsf{U}}$ the sort $\sigma_{\mathsf{U}}(i)$ $(< \lambda)$ of the i-th argument of U. To define positive primitive formulas of signature $\mathbb{S}$ we fix pairwise disjoint countably infinite sets X_μ of variables for each sort $\mu < \lambda$. The atomic formulas are $\mathsf{U}(y_0,\dots,y_{n_{\mathsf{U}}-1})$ and $y =_\mu z$ where U is a relation symbol in $\mathbb{S}$, $y_i \in X_{\sigma_{\mathsf{U}}(i)}$ (i.e., y_i is a variable of sort $\sigma_{\mathsf{U}}(i)$) for each $i < n_{\mathsf{U}}$, and $y, z \in X_\mu$ (i.e., y, z are variables of sort μ). A positive primitive formula of signature $\mathbb{S}$ is an existential formula whose matrix is a conjunction of atomic formulas of signature $\mathbb{S}$.

A multisorted relational structure of signature $\mathbb{S}$ is a pair $\mathbf{B} = \big((B_\mu)_{\mu<\lambda}, (\mathsf{U}^{\mathbf{B}})_{\mathsf{U}\in\mathbb{S}}\big)$ where $(B_\mu)_{\mu<\lambda}$ is a λ-sequence of nonempty sets called the sorts of $\mathbf{B}$, and for each $\mathsf{U} \in \mathbb{S}$, the interpretation $\mathsf{U}^{\mathbf{B}}$ of U in $\mathbf{B}$ is a multisorted relation $\mathsf{U}^{\mathbf{B}} \subseteq \prod_{i<n_{\mathsf{U}}} B_{\sigma_{\mathsf{U}}(i)}$, that is, for each $i < n_{\mathsf{U}}$ the i-th coordinate of each tuple in $\mathsf{U}^{\mathbf{B}}$ is in the $\sigma_{\mathsf{U}}(i)$-th sort. Positive primitive formulas of signature $\mathbb{S}$ can be interpreted in $\mathbf{B}$ the usual way, and each positive primitive formula of signature $\mathbb{S}$ with free variables x_{μ_i,j_i} $(i < l)$ defines a multisorted relation $R \subseteq \prod_{i<l} B_{\mu_i}$. We will say that a multisorted relation $R \subseteq \prod_{i<m} B_{\mu_i}$ on the sorts of $\mathbf{B}$ is p.p. (positive primitive) definable in $\mathbf{B}$ if there is a positive primitive formula of signature $\mathbb{S}$ that defines R.

More informally, by a multisorted relational structure with λ sorts we will mean a pair $\mathbf{B} = \big((B_\mu)_{\mu<\lambda}, \mathcal{S}\big)$ where, as before, $(B_\mu)_{\mu<\lambda}$ is a λ-sequence of sets called the sorts of $\mathbf{B}$, and $\mathcal{S}$ is a family of multisorted relations on those sorts, that is, $\mathcal{S}$ is a family of subsets of finite products of sorts. Now we want to define what an appropriate signature for $\mathbf{B}$ is if we want to index the relations in $\mathcal{S}$. If there is more than one sort, this requires a decision, because a subset of a product $\prod_{i<n} B_{\mu_i}$ may not determine the list of factors B_{μ_i} $(i < n)$ uniquely. To be as inclusive as possible we will say that an appropriate signature for $\mathbf{B} = \big((B_\mu)_{\mu<\lambda}, \mathcal{S}\big)$ is a multisorted relational signature $\mathbb{S}$ with the same number (λ) of sorts as $\mathbf{B}$ such that there is a multisorted relational structure $\mathbf{B}^{\mathbb{S}} = \big((B_\mu)_{\mu<\lambda}, (\mathsf{U}^{\mathbf{B}^{\mathbb{S}}})_{\mathsf{U}\in\mathbb{S}}\big)$ of signature $\mathbb{S}$ on the same sorts such that

- $\mathcal{S} = \{\mathsf{U}^{\mathbf{B}^{\mathbb{S}}} : \mathsf{U} \in \mathbb{S}\}$, and
- for each choice of n, $\sigma\colon n \to \lambda$, and $U \in \mathcal{S}$ with $U \subseteq \prod_{i<n} B_{\sigma(i)}$ the signature $\mathbb{S}$ contains a relation symbol U with $n_{\mathsf{U}} = n$ and $\sigma_{\mathsf{U}} = \sigma$ whose interpretation in $\mathbf{B}^{\mathbb{S}}$ is U.

We will refer to such a $\mathbf{B}^{\mathbb{S}}$ as an indexing of $\mathbf{B}$ with an appropriate signature $\mathbb{S}$.

The multisorted structures we will be concerned with are structures whose sorts are sections (quotients of subalgebras) of a given algebra and whose relations are subalgebras of products of the sort algebras.

Lemma 4.6. *Let $\overline{\mathbf{Q}}_\mu = \mathbf{Q}_\mu/\theta_\mu$ ($\mu < \lambda$) be quotients of subalgebras of a fixed algebra $\mathbf{A}$, let $\overline{\mathcal{S}}$ be a family of subalgebras of finite products of some $\overline{\mathbf{Q}}_\mu$'s, and let $\overline{\mathbf{Q}}^{\mathbb{S}}$ be an indexing of the multisorted relational structure $\overline{\mathbf{Q}} = \big((\overline{Q}_\mu)_{\mu<\lambda}; \overline{\mathcal{S}}\big)$ by an appropriate signature $\mathbb{S}$. For each $\overline{U} := \mathsf{U}^{\overline{\mathbf{Q}}^{\mathbb{S}}} \in \overline{\mathcal{S}}$, let $\mathbf{U}$ be the full inverse image of the subalgebra $\overline{\mathbf{U}}$ of $\prod_{i<n_{\mathsf{U}}} \overline{\mathbf{Q}}_{\sigma_{\mathsf{U}}(i)}$ under the natural homomorphism $\prod_{i<n_{\mathsf{U}}} \mathbf{Q}_{\sigma_{\mathsf{U}}(i)} \to \prod_{i<n_{\mathsf{U}}} \overline{\mathbf{Q}}_{\sigma_{\mathsf{U}}(i)}$, and let $\mathcal{S}$ be the collection of all these U's. If a multisorted relation $\overline{R} \le \prod_{i<m} \overline{Q}_{\tau(i)}$ is p.p. definable in the multisorted relational structure $\overline{\mathbf{Q}}^{\mathbb{S}}$, then*

1. *$\overline{R}$ is the underlying set of a subalgebra $\overline{\mathbf{R}}$ of $\prod_{i<m} \overline{\mathbf{Q}}_{\tau(i)}$, and*
2. *for the subalgebra $\mathbf{R}$ of $\prod_{i<m} \mathbf{Q}_{\tau(i)}$ obtained by taking the full inverse image of $\overline{\mathbf{R}}$ under the natural homomorphism $\prod_{i<m} \mathbf{Q}_{\tau(i)} \to \prod_{i<m} \overline{\mathbf{Q}}_{\tau(i)}$, R is p.p. definable in the (one-sorted) relational structure $(A; \mathcal{S} \cup \{\theta_\mu : \mu < \lambda\})$.*

Proof. Let $\Phi(\mathbf{x})$ be a p.p. formula of signature $\mathbb{S}$ that defines $\overline{R} \le \prod_{i<m} \overline{Q}_{\tau(i)}$ in $\overline{\mathbf{Q}}^{\mathbb{S}}$. Thus $\Phi(\mathbf{x})$ has the form $\exists \mathbf{y}\, \Psi(\mathbf{x}')$ where $\mathbf{x}'$ is the concatenation of $\mathbf{x}$ with $\mathbf{y}$, and $\Psi(\mathbf{x}')$ is a conjunction of atomic formulas of signature $\mathbb{S}$. Let $\mathbf{x}' = (x_i)_{i<n}$ and let $\tau'\colon n \to \lambda$ be defined so that $\tau'(i)$ is the sort of x_i for each $i < n$. Thus $\mathbf{x} = (x_i)_{i<m}$ and $\mathbf{y} = (x_i)_{m\le i<n}$. Since $\Phi(\mathbf{x})$ defines $\overline{R}$, $\tau'(i) = \tau(i)$ for all $i < m$. Hence τ' is an extension of τ.

We may assume without loss of generality that in each conjunct of $\Psi(\mathbf{x}')$ which is a relational atomic formula $\mathsf{U}((x_{i_j})_{j<n_{\mathsf{U}}})$, the variables x_{i_j} ($j < n_{\mathsf{U}}$) are pairwise distinct. For if there is a repetition among the variables, say $x_{i_0} = x_{i_1}$, then $\sigma_{\mathsf{U}}(0) = \sigma_{\mathsf{U}}(1)$, and $\mathsf{U}((x_{i_j})_{j<n_{\mathsf{U}}})$ is logically equivalent to $\exists z\big(\mathsf{U}(z, (x_{i_j})_{1\le j<n_{\mathsf{U}}}) \wedge z =_{\sigma_{\mathsf{U}}(0)} x_{i_1}\big)$ for any variable $z \in X_{\sigma_{\mathsf{U}}(0)}$ not occurring in $\Phi(\mathbf{x})$. Hence the p.p. formula

$$\exists \mathbf{y} z \left(\mathsf{U}(z, (x_{i_j})_{1\le j<n_{\mathsf{U}}}) \wedge z =_{\sigma_{\mathsf{U}}(0)} x_{i_1} \wedge \bigwedge(\text{all other conjuncts of } \Psi(\mathbf{x}'))\right)$$

defines the same relation in $\overline{\mathbf{Q}}^{\mathbb{S}}$ as $\Phi(\mathbf{x})$, and has fewer repetitions of variables in relational atomic formulas. Repeating this precedure we can eliminate all repetitions of variables in relational atomic formulas.

Now, after this adjustment, for each conjunct C of $\Psi(\mathbf{x}')$ let $C^*(\mathbf{x}')$ denote the formula

$$C \wedge \bigwedge_{i<n} x_i =_{\tau'(i)} x_i.$$

Let $\Psi^*(\mathbf{x}')$ denote the conjunction of all the $C^*(\mathbf{x}')$'s obtained in this way, which we will refer to as the conjuncts of $\Psi^*(\mathbf{x}')$. Furthermore, let $\Phi^*(\mathbf{x})$ denote the p.p. formula $\exists\mathbf{y}\,\Psi^*(\mathbf{x}')$. It is clear from this construction that $\Psi^*(\mathbf{x}')$ and $\Psi(\mathbf{x}')$ define the same relation in $\overline{\mathbf{Q}}^{\mathbb{S}}$, and hence so do $\Phi^*(\mathbf{x})$ and $\Phi(\mathbf{x})$. Thus $\Phi^*(\mathbf{x})$ is another p.p. formula that defines $\overline{R}$. We will use this formula to prove (1) and (2).

First we will argue that the sets $\overline{X}_C$ defined by the conjuncts $C^*(\mathbf{x}')$ of $\Psi^*(\mathbf{x}')$ are underlying sets of subalgebras $\overline{\mathbf{X}}_C$ of $\prod_{i<n}\overline{\mathbf{Q}}_{\tau'(i)}$. If C has the form $\mathsf{U}((x_{i_j})_{j<n_{\mathsf{U}}})$, where U is a relation symbol in $\mathbb{S}$, then $\overline{X}_C$ consists of all tuples $(a_i)_{i<n}\in\prod_{i<n}\overline{Q}_{\tau'(i)}$ for which $(a_{i_j})_{j<n_{\mathsf{U}}}\in\mathsf{U}^{\overline{\mathbf{Q}}^{\mathbb{S}}}$. Since the variables x_{i_j} $(j<n_{\mathsf{U}})$ are pairwise distinct, we get that, up to a permutation of coordinates, $\overline{X}_C$ is the direct product of the algebra $\mathsf{U}^{\overline{\mathbf{Q}}^{\mathbb{S}}}$ with all algebras $\overline{\mathbf{Q}}_{\tau'(i')}$, $i'\neq i_j$ $(j<n_{\mathsf{U}})$. Hence $\overline{X}_C$ is (the underlying set of) a subalgebra of $\prod_{i<n}\overline{\mathbf{Q}}_{\tau'(i)}$. Similarly, if C has the form $x_j =_{\tau'(j)} x_{j'}$, then $\overline{X}_C$ consists of all tuples $(a_i)_{i<n}\in\prod_{i<n}\overline{Q}_{\tau'(i)}$ for which $a_j=a_{j'}$. It is clear that in this case $\overline{X}_C$ is (the underlying set of) a subalgebra of $\prod_{i<n}\overline{\mathbf{Q}}_{\tau'(i)}$. The set defined by $\Psi^*(\mathbf{x}')$, the conjunction of the $C^*(\mathbf{x}')$'s, is the intersection of the algebras $\overline{\mathbf{X}}_C$, and is therefore a subalgebra $\overline{\mathbf{Y}}$ of $\prod_{i<n}\overline{\mathbf{Q}}_{\tau'(i)}$. Finally, the set defined by $\Phi^*(\mathbf{x})=\exists\mathbf{y}\,\Psi^*(\mathbf{x}')$ is the projection of the algebra $\overline{\mathbf{Y}}$ onto its first m coordinates, and is therefore a subalgebra of $\prod_{i<m}\overline{\mathbf{Q}}_{\tau(i)}$. This completes the proof of (1).

To prove (2) let $\mathbb{S}^-$ denote the one-sorted relational signature obtained from $\mathbb{S}$ by forgetting the sorts; that is, $\mathbb{S}^-$ and $\mathbb{S}$ have the same relation symbols, and each relation symbol U has the same arity n_{U} in $\mathbb{S}^-$ and $\mathbb{S}$, the only difference is that in $\mathbb{S}^-$ there is no need for a sort function σ_{U}. We will expand $\mathbb{S}^-$ by adding λ new binary relation symbols Θ_μ $(\mu<\lambda)$, and denote the resulting one-sorted relational signature by $\mathbb{T}$. Clearly, the relational atomic formulas of signature $\mathbb{S}$ are also relational atomic formulas of signature $\mathbb{T}$. Therefore, by replacing each equational atomic formula $x_j =_{\tau'(j)} x_{j'}$ by $\Theta_{\tau'(j)}(x_j,x_{j'})$ we can associate a p.p. formula of signature $\mathbb{T}$ to each multisorted p.p. formula of signature $\mathbb{S}$. We will denote the formulas of signature $\mathbb{T}$ associated to $\Phi^*(\mathbf{x})$, $\Psi^*(\mathbf{x}')$, and each conjunct $C^*(\mathbf{x}')$ of $\Psi^*(\mathbf{x}')$ by $\Phi^\dagger(\mathbf{x})$, $\Psi^\dagger(\mathbf{x}')$, and $C^\dagger(\mathbf{x}')$, respectively.

It is clear from the definition of $\mathcal{S}$ that $\mathbb{T}$ is an appropriate signature for the one-sorted relational structure $\mathbf{P}:=(A;\mathcal{S}\cup\{\theta_\mu:\mu<\lambda\})$; namely we get an indexing $\mathbf{P}^{\mathbb{T}} := \big(A;(\mathsf{U}^{\mathbf{P}^{\mathbb{T}}})_{\mathsf{U}\in\mathbb{S}^-},(\Theta_\mu^{\mathbf{P}^{\mathbb{T}}})_{\mu<\lambda}\big)$ of $\mathbf{P}$ by $\mathbb{T}$ if for each $\mathsf{U}\in\mathbb{S}^-$ we define $\mathsf{U}^{\mathbf{P}^{\mathbb{T}}}$ to be the full inverse image of $\mathsf{U}^{\overline{\mathbf{Q}}^{\mathbb{S}}}$ under the natural homomorphism $\prod_{i<n_{\mathsf{U}}}\mathbf{Q}_{\sigma_{\mathsf{U}}(i)}\to$

$\prod_{i<n_{\mathsf{U}}} \overline{\mathbf{Q}}_{\sigma_{\mathsf{U}}(i)}$, and for each $\mu < \lambda$, we define $\Theta_\mu^{\mathbf{P}^{\mathbb{T}}}$ to be the congruence θ_μ of $\mathbf{Q}_\mu$. To prove (2), we will argue that $\Phi^\dagger(\mathbf{x})$ defines the relation R in $\mathbf{P}^{\mathbb{T}}$.

Each conjunct $C^\dagger(\mathbf{x}')$ of $\Psi^\dagger(\mathbf{x}')$ has the form

$$C^\dagger \wedge \bigwedge_{i<n} \Theta_{\tau'(i)}(x_i, x_i)$$

where $C^\dagger$ is the same as C if C is a relational atomic formula $\mathsf{U}((x_{i_j})_{j<n_{\mathsf{U}}})$, and $C^\dagger$ is $\Theta_{\tau'(j)}(x_j, x_{j'})$ if C is an equational atomic formula of the form $x_j =_{\tau'(j)} x_{j'}$. In either case, let X_C denote the relation defined by $C^\dagger(\mathbf{x}')$ in $\mathbf{P}^{\mathbb{T}}$. In the first case X_C consists of all tuples $(a_i)_{i<n} \in A^n$ that satisfy the following conditions: (i) $(a_{i_j})_{j<n_{\mathsf{U}}} \in \mathsf{U}^{\mathbf{P}^{\mathbb{T}}}$, and (ii) $(a_i, a_i) \in \theta_{\tau'(i)}$ for all $i < n$, that is, $a_i \in Q_{\tau'(i)}$ for all $i < n$. Thus X_C consists of all tuples $(a_i)_{i<n} \in \prod_{i<n} Q_{\tau'(i)}$ for which $(a_{i_j})_{j<n_{\mathsf{U}}} \in \mathsf{U}^{\mathbf{P}^{\mathbb{T}}}$. Since $\mathsf{U}^{\mathbf{P}^{\mathbb{T}}}$ is the full inverse image of $\mathsf{U}^{\overline{\mathbf{Q}}^{\mathbb{S}}}$ under the natural homomorphism $\prod_{i<n_{\mathsf{U}}} \mathbf{Q}_{\sigma_{\mathsf{U}}(i)} \to \prod_{i<n_{\mathsf{U}}} \overline{\mathbf{Q}}_{\sigma_{\mathsf{U}}(i)}$, we see from the earlier description of $\overline{X}_C$ that X_C is the full inverse image of $\overline{X}_C$ under the natural homomorphism $\nu'\colon \prod_{i<n} \mathbf{Q}_{\tau'(i)} \to \prod_{i<n} \overline{\mathbf{Q}}_{\tau'(i)}$. The relation Y defined in $\mathbf{P}^{\mathbb{T}}$ by the conjunction $\Psi^\dagger(\mathbf{x}')$ of the $C^\dagger(\mathbf{x}')$'s is the intersection of the X_C's. Since each X_C is the full inverse image of $\overline{X}_C$ under ν', their intersection Y is the full inverse image of the intersection $\overline{Y}$ of the $\overline{X}_C$'s under ν'. Finally, the relation defined by $\Phi^\dagger(\mathbf{x}')$ in $\mathbf{P}^{\mathbb{T}}$ is the projection $\mathrm{pr}_I Y$ of Y onto the set $I = m = \{0, \ldots, m-1\}$ of first m coordinates. Since Y is the full inverse image of $\overline{Y}$ under the natural homomorphism $\nu'\colon \prod_{i<n} \mathbf{Q}_{\tau'(i)} \to \prod_{i<n} \overline{\mathbf{Q}}_{\tau'(i)}$, $\mathrm{pr}_I Y$ is the full inverse image of $\mathrm{pr}_I \overline{Y}$ under the natural homomorphism $\nu\colon \prod_{i<m} \mathbf{Q}_{\tau(i)} \to \prod_{i<m} \overline{\mathbf{Q}}_{\tau(i)}$. We have shown earlier that $\mathrm{pr}_I \overline{Y} = \overline{R}$. Therefore the relation defined by $\Phi^\dagger(\mathbf{x}')$ in $\mathbf{P}^{\mathbb{T}}$ is the full inverse image R of $\overline{R}$ under ν, as claimed. $\square$

After these preparations we return to the proof of Theorem 4.3.

Proof of Theorem 4.3. Let us fix k and A. The statement is trivial for $|A| = 1$, so we will assume that $|A| \geq 2$. Let $c_0 = |A|^{|A|+1}\big(B(|A|+1) - 1\big)$ where $B(n)$ denotes the n-th Bell number, and let $c = \max(k, c_0)$. Notice that since the function assigning to each equivalence relation on $\{0, 1, \ldots, n\}$ the equivalence relation obtained by omitting the equivalence class of n establishes a one-to-one correspondence between the equivalence relations on an $(n+1)$-element set and the equivalence relations on all subsets of an n-element set, the factor $B(|A|+1) - 1$ in c_0 is the number of equivalence relations on all nonempty subsets of A.

Let $\mathcal{C}$ be a clone on A such that the algebra $\mathbf{A} := (A; \mathcal{C})$ has a k-parallelogram term and generates a residually small variety $\mathcal{V}$. Let $\mathcal{R} := \mathcal{C}^\perp$ denote the relational clone of $\mathbf{A}$, that is, $\mathcal{R}$ is the set of all (finitary) compatible relations of $\mathbf{A}$, and let $\mathcal{R}^{(\leq c)}$ consist of all relations of arity $\leq c$ in $\mathcal{R}$. Our goal is to show that $\mathcal{C} = (\mathcal{R}^{(\leq c)})^\perp$. Since there are only finitely many relations of arity $\leq c$ on A and c is independent of the choice of the clone $\mathcal{C}$, this will imply that there are only finitely many choices for $\mathcal{C}$, and hence will complete the proof of Theorem 4.3.

By Corollary 4.5 the equality $\mathcal{C} = (\mathcal{R}^{(\leq c)})^{\perp}$ to be proved is equivalent to the condition that every critical relation of $\mathbf{A}$ is p.p. definable in the relational structure $(A; \mathcal{R}^{(\leq c)})$. Let R be a critical relation of $\mathbf{A}$ of arity $\ell > c$. Since $c \geq \max(k, 2^3)$, we get from Theorem 4.1 that the reduction $\overline{R}$ of R has the structure described in Theorem 2.5. We will use the notation of Theorem 2.5 (with the exception that k is replaced by ℓ). Thus $\overline{\mathbf{R}}$ is a subdirect product of some sections $\overline{\mathbf{A}}_i = \mathbf{A}_i/\theta_i$ of $\mathbf{A}$, and $\mathbf{R}$ is the full inverse image of $\overline{\mathbf{R}}$ under the natural homomorphism $\prod_{i<\ell} \mathbf{A}_i \to \prod_{i<\ell} \overline{\mathbf{A}}_i$. Let $\lambda = \bigl|\{\overline{\mathbf{A}}_i : i < \ell\}\bigr|$, let $\overline{\mathbf{Q}}_\mu = \mathbf{Q}_\mu/\theta_\mu$ $(\mu < \lambda)$ be an enumeration of $\{\overline{\mathbf{A}}_i : i < \ell\}$, and let $\tau\colon \ell \to \lambda$ denote the unique function such that $\overline{\mathbf{A}}_i = \overline{\mathbf{Q}}_{\tau(i)}$ for all $i < \ell$. Furthermore, let $\overline{\mathcal{S}}$ be the collection of all subalgebras of products of $\leq c$ algebras of the form $\overline{\mathbf{Q}}_\mu$ $(\mu < \lambda)$. Then $\overline{\mathbf{Q}} := \bigl((\overline{Q}_\mu)_{\mu<\lambda}; \overline{\mathcal{S}}\bigr)$ is a multisorted relational structure with λ sorts where $\lambda \leq B(|A|+1) - 1$. Let $\overline{\mathbf{Q}}^{\mathbb{S}}$ be an indexing of $\overline{\mathbf{Q}}$ by an appropriate signature $\mathbb{S}$, and define $\mathcal{S}$ as described in Lemma 4.6. By construction, $\mathcal{S} \subseteq \mathcal{R}^{(\leq c)}$. Therefore the statement we want to prove, namely that R is p.p. definable in $(A; \mathcal{R}^{(\leq c)})$, will follow from Lemma 4.6 if we prove that $\overline{R}$ is p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$.

So, it remains to show that $\overline{R}$ is p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$. Recall that $\ell > c \geq \max(k, 8)$. Hence $\ell \geq \max(k, 3)$, and therefore by Theorem 4.1, the sections $\overline{\mathbf{A}}_i = \overline{\mathbf{Q}}_{\tau(i)}$ of $\mathbf{A}$ that appear in the subdirect decomposition of $\overline{\mathbf{R}}$ are subdirectly irreducible, and have abelian monoliths $\overline{\mu}_i$ $(i < \ell)$. Since $\mathcal{V}$ is congruence modular by Theorems 3.3 (2) and 3.5, and is residually small by assumption, we get from 10.15 of [5] that the centralizer $\overline{\rho}_i$ of $\overline{\mu}_i$ is abelian for each $i < \ell$. Let $\overline{\rho}$ be the product congruence $\overline{\rho}_0 \times \cdots \times \overline{\rho}_{\ell-1}$ of $\overline{\mathbf{R}}$, and let t be the number of $\overline{\rho}$-classes of $\overline{\mathbf{R}}$. Choose elements $\mathbf{o}^{(m)}$ $(m < t)$ of $\overline{\mathbf{R}}$ that form a transversal for the $\overline{\rho}$-classes, and let $\mathbf{o}^{(m)} = (o_i^{(m)})_{i<\ell}$ where $o_i^{(m)} \in \overline{\mathbf{A}}_i$ for all $m < t$ and $i < \ell$. Since condition (8) of Theorem 2.5 holds for $\overline{\mathbf{R}}$, it follows that the elements $o_i^{(m)}$ $(m < t)$ form a transversal for the $\overline{\rho}_i$-classes of $\overline{\mathbf{A}}_i$ for each $i < \ell$. Let $\overline{\mathbf{O}}$ denote the subalgebra of $\overline{\mathbf{R}}$ generated by the elements $\mathbf{o}^{(m)}$ $(m < t)$.

Let d be a difference term for $\mathcal{V}$. Since the congruence $\overline{\rho}_i$ of $\overline{\mathbf{A}}_i$ $(i < \ell)$ is abelian, 5.7 of [5] implies that

$$\overline{M}_i := \{(x, y, z, d(x, y, x)) : x, y, z \in \overline{A}_i \text{ with } x\,\overline{\rho}_i\,y\,\overline{\rho}_i\,z\}$$

is the underlying set of a subalgebra $\overline{\mathbf{M}}_i$ of $\overline{\mathbf{A}}_i^4$ for each $i < \ell$. Moreover, $d(x, x, z) = z$ and $d(x, z, z) = x$ hold for all elements $x, z \in \overline{A}_i$ with $x\,\overline{\rho}_i\,z$.

Claim 4.7. *Let* $\overline{\mathbf{U}}$ *and* $\overline{\mathbf{V}}$ *be arbitrary subalgebras of* $\overline{\mathbf{R}}$ *that contain* $\overline{\mathbf{O}}$. *If* $\overline{U}, \overline{V} \subseteq \prod_{i<\ell} \overline{Q}_{\tau(i)} \left(= \prod_{i<\ell} \overline{A}_i\right)$ *are p.p. definable in* $\overline{\mathbf{Q}}^{\mathbb{S}}$, *then so is*

$$\overline{U} \boxplus \overline{V} := \{\mathbf{w} \in \overline{R} : \mathbf{w} = d(\mathbf{u}, \mathbf{o}, \mathbf{v}) \text{ for some } \mathbf{u} \in \overline{U}, \mathbf{o} \in \overline{O}, \mathbf{v} \in \overline{V} \text{ with } \mathbf{u}\,\overline{\rho}\,\mathbf{o}\,\overline{\rho}\,\mathbf{v}\},$$

and $\overline{U} \boxplus \overline{V}$ *is the underlying set of a subalgebra* $\overline{\mathbf{U}} \boxplus \overline{\mathbf{V}}$ *of* $\overline{\mathbf{R}}$ *that contains both* $\overline{\mathbf{U}}$ *and* $\overline{\mathbf{V}}$.

To prove Claim 4.7 let $\mathbf{x} = (x_i)_{i<\ell}$, $\mathbf{y} = (y_i)_{i<\ell}$, $\mathbf{z} = (z_i)_{i<\ell}$, and $\mathbf{w} = (w_i)_{i<\ell}$ be disjoint sets of variables such that $x_i, y_i, z_i, w_i \in X_{\tau(i)}$ for each $i < \ell$. Assume that $\overline{U}$, $\overline{V}$, and $\overline{O}$, are defined in $\overline{\mathbf{Q}}^{\mathbb{S}}$ by the p.p. formulas $\Phi(\mathbf{x})$, $\Psi(\mathbf{y})$, and $\Omega(\mathbf{z})$, respectively. For each $i < \ell$ we have $\overline{M}_i \in \overline{\mathcal{S}}$, because $\overline{\mathbf{M}}_i$ is a subalgebra of $\overline{\mathbf{Q}}^4_{\tau(i)}$ $(= \overline{\mathbf{A}}^4_i)$ and $c \geq 4$. So our choice of $\mathbb{S}$ ensures that $\overline{M}_i$ is the interpretation in $\overline{\mathbf{Q}}^{\mathbb{S}}$ of a 4-ary relation symbol M_i whose variables are all of sort $\tau(i)$. Hence the definitions of $\overline{U} \boxplus \overline{V}$ and $\overline{M}_i$ $(i < \ell)$ immediately imply that $\overline{U} \boxplus \overline{V}$ is defined by the p.p. formula

$$\Gamma(\mathbf{w}) := \exists \mathbf{x}\, \exists \mathbf{z}\, \exists \mathbf{y} \left(\Phi(\mathbf{x}) \wedge \Omega(\mathbf{z}) \wedge \Psi(\mathbf{y}) \wedge \bigwedge_{i<t} \mathsf{M_i}(x_i, z_i, y_i, w_i)\right).$$

Now statement (1) in Lemma 4.6 implies that $\overline{U} \boxplus \overline{V}$ is the underlying set of a subalgebra $\overline{\mathbf{U}} \boxplus \overline{\mathbf{V}}$ of $\overline{\mathbf{R}}$. To show that $\overline{U} \subseteq \overline{U} \boxplus \overline{V}$ let $\mathbf{u} \in \overline{U}$, and let $\mathbf{o}^{(m)}$ be the transversal element chosen from the $\overline{\rho}$-class of $\mathbf{u}$. By assumption $\mathbf{o}^{(m)} \in \overline{O} \subseteq \overline{V}$, therefore $\mathbf{u} = d(\mathbf{u}, \mathbf{o}^{(m)}, \mathbf{o}^{(m)}) \in \overline{U} \boxplus \overline{V}$. The proof that $\overline{V} \subseteq \overline{U} \boxplus \overline{V}$ is similar. This completes the proof of Claim 4.7.

Next we want to show that every subalgebra of $\overline{\mathbf{R}}$ with a small generating set has at most c essential coordinates; all other coordinates are repetitions of essential coordinates.

Claim 4.8. *If a subalgebra* $\overline{\mathbf{U}}$ *of* $\overline{\mathbf{R}}$ *is generated by* $t + 1$ *elements, then there exist* $I \subseteq \{0, 1, \ldots, \ell - 1\}$ *and* $\varphi\colon \{0, 1, \ldots, \ell - 1\} \to I$ *such that* $|I| \leq c$, $\varphi(i) = i$ *for all* $i \in I$, $\tau(i) = \tau(\varphi(i))$ *for all* $i < \ell$, *and*

$$(x_i)_{i<\ell} \in \overline{\mathbf{U}} \quad \Rightarrow \quad x_i = x_{\varphi(i)} \text{ for all } i < \ell.$$

To prove the claim let $\mathbf{a}_n := (a_{ni})_{i<\ell} \in \overline{\mathbf{R}}$ $(n \leq t)$ be $t + 1$ elements that generate $\overline{\mathbf{U}}$. Define an equivalence relation $\sim$ on $\{0, 1, \ldots, \ell - 1\}$ as follows: for $i, j < \ell$ let $i \sim j$ iff $\tau(i) = \tau(j)$ and in each generator $\mathbf{a}_n$ the i-th and j-th coordinates $a_{ni}, a_{nj} \in \overline{Q}_{\tau(i)}$ $(= \overline{A}_i = \overline{A}_j)$ are equal. The range of τ is λ, and we have established earlier that $\lambda \leq B(|A| + 1) - 1$. Furthermore, for each $i < \ell$ the $(t+1)$-tuple $(a_{ni})_{n\leq t}$ of i-th coordinates of the selected generators of $\overline{\mathbf{U}}$ is an element of $\overline{\mathbf{Q}}^{t+1}_{\tau(i)}$, so since $|\overline{\mathbf{Q}}_{\tau(i)}| \leq |A|$ and $t \leq |A|$, there are at most $|A|^{|A|+1}$ choices for $(a_{ni})_{n\leq t}$. Therefore $\sim$ has at most $\big(B(|A| + 1) - 1\big)|A|^{|A|+1} = c_0$ equivalence classes. Now choose a transversal I for the equivalence classes of $\sim$, and let $\varphi\colon \{0, 1, \ldots, \ell - 1\} \to I$ assign to each $i < \ell$ the transversal element in $i/{\sim}$. It is clear from this construction that $|I| \le c_0 \le c$, $\varphi(i) = i$ for all $i \in I$, and $\tau(\varphi(i)) = \tau(i)$ for all $i < \ell$. Moreover, since $i \sim \varphi(i)$ for all $i < \ell$, we get that in each generator $\mathbf{a}_n$ $(n \le t)$ of $\overline{\mathbf{U}}$ the i-th and $\varphi(i)$-th coordinates are equal. Therefore the same holds for all elements of $\overline{\mathbf{U}}$ as well. This completes the proof of Claim 4.8.

Claim 4.9. *$\overline{R} \subseteq \prod_{i<\ell} \overline{Q}_{\tau(i)}$ is p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$.*

For each element $\mathbf{r} \in \overline{\mathbf{R}}$ let $\overline{\mathbf{U}}_\mathbf{r}$ denote the subalgebra of $\overline{\mathbf{R}}$ generated by the $t+1$ elements $\mathbf{o}^{(m)}$ $(m < t)$ and $\mathbf{r}$. By Claim 4.8 there exist $I_\mathbf{r} \subseteq \{0, 1, \dots, \ell - 1\}$ and $\varphi_\mathbf{r} \colon \{0, 1, \dots, \ell - 1\} \to I_\mathbf{r}$ such that $|I_\mathbf{r}| \le c$, $\varphi_\mathbf{r}(i) = i$ for all $i \in I_\mathbf{r}$, $\tau(i) = \tau(\varphi_\mathbf{r}(i))$ for all $i < \ell$, and $x_i = x_{\varphi_\mathbf{r}(i)}$ for all $(x_i)_{i<\ell} \in \overline{\mathbf{U}}_\mathbf{r}$ and $i < \ell$. Thus $\overline{\mathbf{W}}_\mathbf{r} := \mathrm{pr}_{I_\mathbf{r}} \overline{\mathbf{U}}_\mathbf{r}$ is a subalgebra of $\prod_{i \in I_\mathbf{r}} \overline{\mathbf{Q}}_{\tau(i)}$ $\Big(= \prod_{i \in I_\mathbf{r}} \overline{\mathbf{A}}_i\Big)$, hence $\overline{W}_\mathbf{r} \in \overline{\mathcal{S}}$. Moreover, $\overline{U}_\mathbf{r}$ is p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$ by the formula

$$\Psi_\mathbf{r}(\mathbf{x}) := \mathsf{W}_\mathbf{r}\big((x_i)_{i \in I_\mathbf{r}}\big) \wedge \bigwedge_{i<\ell,\ i \notin I_\mathbf{r}} x_i =_{\tau(i)} x_{\varphi_\mathbf{r}(i)}$$

where $\mathbf{x} = (x_i)_{i<l}$ is a sets of distinct variables such that $x_i \in X_{\tau(i)}$ for each $i < \ell$, and $\mathsf{W}_\mathbf{r}$ is a relation symbol in $\mathbb{S}$ such that $n_{\mathsf{W}_\mathbf{r}} = |I_\mathbf{r}|$, $\sigma_{\mathsf{W}_\mathbf{r}} \colon I_\mathbf{r} \to \lambda$ with $\sigma_{\mathsf{W}_\mathbf{r}}(i) = \tau(i)$ for all $i \in I_\mathbf{r}$, and the interpretation of $\mathsf{W}_\mathbf{r}$ in $\overline{\mathbf{Q}}^{\mathbb{S}}$ is $\overline{\mathbf{W}}_\mathbf{r}$.

By construction, each $\overline{\mathbf{U}}_\mathbf{r}$ $(\mathbf{r} \in \overline{\mathbf{R}})$ contains $\overline{\mathbf{O}}$, and for $\mathbf{r} = \mathbf{o}^{(0)}$ we have that $\overline{\mathbf{O}} = \overline{\mathbf{U}}_\mathbf{r}$. Thus $\overline{O}$ and all $\overline{U}_\mathbf{r}$ $(\mathbf{r} \in \overline{\mathbf{R}})$ are p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$. Therefore a repeated application of Claim 4.7 yields that

$$\boxplus_{\mathbf{r} \in \overline{\mathbf{R}}} \overline{\mathbf{U}}_\mathbf{r} = \overline{\mathbf{R}},$$

and that $\overline{R}$ is p.p. definable in $\overline{\mathbf{Q}}^{\mathbb{S}}$. This completes the proof of the claim and of Theorem 4.3. □

A. Bulatov proved in [3] that there are only finitely many distinct clones on $\{0, 1, 2\}$ that contain a Maltsev operation. (In fact, he proved that there are 1129 such clones.) The finiteness of this set of clones follows easily from Theorem 4.3, as does the more general result involving parallelogram operations.

Corollary 4.10. *For a fixed integer $k > 1$ there are only finitely many distinct clones on the set $\{0, 1, 2\}$ that have a k-parallelogram operation.*

Proof. Every algebra of fewer than four elements in a congruence modular variety generates a residually small subvariety. (This fact follows from the Freese-McKenzie characterization of finitely generated congruence modular varieties in [4].) □

We have to fix k in the previous theorem, since for each $k > 1$ there exist clones on $\{0, 1, 2\}$ that contains $(k + 1)$-nu operations but no k-nu operations (so there are infinitely many distinct clones with parallelogram operations on $\{0, 1, 2\}$). This follows from the fact that there exists such operations on $\{0, 1\}$.

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(Keith Kearnes) Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, U.S.A.

E-mail address: `Keith.Kearnes@Colorado.EDU`

(Ágnes Szendrei) Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, U.S.A., and Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H–6720 Szeged, Hungary

E-mail address: `szendrei@euclid.colorado.edu, a.szendrei@math.u-szeged.hu`