RESIDUAL SMALLNESS AND WEAK CENTRALITY

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Abstract. We develop a method of creating skew congruences on subpowers of finite algebras using groups of twin polynomials, and apply it to the investigation of residually small varieties generated by nilpotent algebras. We prove that a residually small variety generated by a finite nilpotent (in particular, a solvable E-minimal) algebra is weakly abelian. Conversely, we show in two special cases that a weakly abelian variety is residually bounded by a finite number: when it is generated by an E-minimal, or by a finite strongly nilpotent algebra. This establishes the RS-conjecture for E-minimal algebras.

1. Introduction

One of the main areas of investigation in the theory of general algebraic structures is to describe the subdirectly irreducible algebras in the variety $\mathsf{V}(\mathbf{A})$ generated by a finite algebra $\mathbf{A}$. For most finite algebras $\mathbf{A}$ there are only two, sharply contrasting possibilities for this class of subdirectly irreducibles. It either consists of finitely many finite algebras only (in this case we say that $\mathsf{V}(\mathbf{A})$ is residually bounded by a finite number), or it is a proper class, having members of unbounded cardinality (that is, $\mathsf{V}(\mathbf{A})$ is residually large). Those finite algebras that exhibit this behavior are said to satisfy the RS-conjecture.

It is very difficult to construct finite algebras that do not satisfy the RS-conjecture (see [9], [11]). A celebrated result of Ralph Freese and Ralph McKenzie [2] proves the RS-conjecture for every finite algebra $\mathbf{A}$ in a congruence modular variety. This result also tells us the residual character of $\mathbf{A}$. If a certain "bad" situation occurs in $\mathbf{A}$, then arbitrarily large subdirectly irreducibles are constructed in $\mathsf{V}(\mathbf{A})$. If this "bad" situation does not occur, then a finite bound is established for these subdirectly irreducibles. For example, if $\mathbf{A}$ is a group, then the "bad" situation is the existence of a nonabelian Sylow-subgroup of $\mathbf{A}$. Even in the general modular case, the presence of a nilpotent, but nonabelian congruence on a subalgebra of $\mathbf{A}$ is such a "bad" situation.

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All presently known constructions of finite algebras not satisfying the RS-conjecture involve the existence of a nonabelian prime quotient. Therefore, in the light of the Freese-McKenzie theorem, it is natural to ask: which finite nilpotent algebras satisfy the RS-conjecture? Is it true that if such an algebra generates a residually small variety, then it has to be abelian in some sense?

An affirmative answer to a variant of this question has been given in [5]. In that paper the strong abelian property (also discovered by McKenzie) is considered instead of the "normal" version. The nilpotence concept stemming from strong abelianness is called strong nilpotence. In [5] it is shown that strongly nilpotent algebras satisfy the RS-conjecture. It is also shown that the variety generated by a finite strongly nilpotent algebra is residually small if and only if every member of this variety is "abelian", but this is neither "normal", nor strong abelianness, but, rather, a new concept called *rectangular* abelianness, or simply rectangularity. Rectangularity is weaker than strong abelianness, but the nilpotence concept coming from rectangularity is the same as strong nilpotence. Therefore we can phrase this result more elegantly by saying that a rectangularly nilpotent algebra generates a residually small variety if and only if this variety is rectangularly abelian, and if so, it is residually bounded by a finite number.

A similar phenomenon occurs when we go back to "normal" nilpotence. It is not true that a residually small variety generated by a finite nilpotent algebra is abelian. However, we shall prove that every member of such a variety is weakly abelian. The weak abelian property is a fourth concept, discovered in [4], that is weaker than both "normal" and rectangular abelianness. The relationship between these four concepts can be summarized by the following diagram:

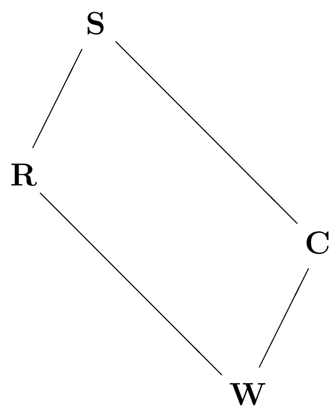

FIGURE 1: **C** = normal, **W** = weak, **R** = rectangular, **S** = strong.

We shall prove that, similarly to the "strong" case, even weak nilpotence (coming from weak abelianness) and residual smallness imply that the variety is weakly abelian.

The analogy with the "strong" case breaks when we consider the converse of this result. It is not true that weakly (or even normally) abelian, finitely generated varieties are residually small. The paper [6] gives a characterization of residual smallness for finitely generated, normally abelian varieties. However, the converse, and hence the RS-conjecture does hold for a special class of nilpotent algebras, the so-called E-minimal algebras. The concept of E-minimality comes from tame congruence theory. The structure of E-minimal algebras has been described in [3] and in [7]. With the exception of two-element, non-solvable algebras, E-minimal algebras are always nilpotent.

Thus, to summarize the main results of the paper, we shall first prove in Theorem 3.1 that if $\mathbf{A}$ is a finite algebra in a residually small variety, then every weakly nilpotent congruence (moreover, tolerance) of $\mathbf{A}$ is weakly abelian. If, in particular, $\mathbf{A}$ itself is nilpotent, then every member of $\mathsf{V}(\mathbf{A})$ is weakly abelian. Conversely, in Theorem 5.4 we show that if $\mathbf{A}$ is an E-minimal algebra in a weakly abelian variety, then $\mathsf{V}(\mathbf{A})$ is residually bounded by a finite number. A similar conclusion holds for strongly nilpotent algebras, as shown by Corollary 4.6.

Here is how the paper is organized. The first main result, Theorem 3.1 is proved in Section 3. It is based on two constructions of large subdirectly irreducible algebras. One of these constructions is inherited from [5], the other one is new, but its origins are in the proof of the Freese-McKenzie Theorem. To prove the second main result we first have to establish some properties of weakly abelian varieties, this is done in Section 4. The basic observation is Lemma 4.4, which has a consequence of independent interest stated in Corollary 4.5: in a weakly abelian variety, every strongly nilpotent tolerance of every finite algebra is rectangular. Combined with the results of [5] this corollary implies that the weak abelian property is equivalent to residual smallness for strongly nilpotent varieties (see Corollary 4.6). The main application of Lemma 4.4 is Corollary 4.9. This result shows that every E-minimal algebra in a weakly abelian variety has a congruence ρ such that the blocks of ρ can be considered as abelian groups, and the factor modulo ρ is essentially unary. This structure is then used in Section 5 to give a finite bound on the size of the algebra, in case it is subdirectly irreducible (see Lemma 5.2).

We shall now give a non-technical outline of the ideas used in the paper. In Section 3 we use the assumption that $\mathsf{V}(\mathbf{A})$ is residually small by trying to construct large subdirectly irreducibles in $\mathsf{V}(\mathbf{A})$. In Section 4 we use the condition that $\mathsf{V}(\mathbf{A})$ is weakly abelian by trying to construct algebras in $\mathsf{V}(\mathbf{A})$ that are not weakly abelian. In both cases we have to construct congruences on subpowers of $\mathbf{A}$.

To do so, we use one of the main ideas of tame congruence theory. Take a subset N of an algebra $\mathbf{A}$ that is the range of an idempotent unary polynomial of $\mathbf{A}$ (or the intersection of such a range with a congruence class of $\mathbf{A}$; such subsets are called E-traces). Consider the unary functions of N induced by the unary polynomials of **A** under which N is closed. It is easy to show that if a partition on N is preserved by all these functions, then it can be extended to a congruence of **A**. Usually we choose the set N so small that these restrictions are either permutations of N, or are constant on N. Such subsets are called permutational. In this case, the induced unary polynomials from a group $\mathbf{G}(N)$, and the invariant partitions can be described using the structure of this group (see Lemma 2.2).

To construct congruences on a subalgebra **B** of $\mathbf{A}^k$ we consider the set $N^k \subseteq A^k$, and try to apply the technique described in the previous paragraph for this subset. For the sake of simplicity let us assume that $k = 2$, and that $B = T$ is a tolerance of **A**. In order for this technique to work, we have to make sure that $N^2 \subseteq T$ is still a permutational subset of **T**. This is ensured by centrality. If we assume that T weakly centralizes N^2, then this condition will be satisfied. Therefore, if **A** is nilpotent in a weak sense, then even $T = A^2$ will work, provided that N is any trace for any minimal quotient of **A** (in the sense of tame congruence theory).

Even if we have the appropriate centrality, we have to be able to compute the group of unary permutations on N^2. This is done by considering twin unary permutations of N. Two unary polynomials are called T-twins if they come from the same term by substituting different, but T-related parameters. If T weakly centralizes N^2, and N is permutational, then the unary polynomials induced by **T** on N^2 are either constant, or are pairs of twin unary polynomial-permutations of N. The twin relation is determined by the set of the twins of the identity map of N, which is a normal subgroup in $\mathbf{G}(N)$. This is called the T-twin group on N.

The technique just described is summarized in Lemma 2.6. However, the twin group has another important use in the paper. In a modular variety, the blocks of abelian congruences can be considered as abelian groups by modular commutator theory (see [1]). These groups also come up on type **2** traces in tame congruence theory. But nilpotent algebras can have type **1** quotients as well, when the traces are essentially unary algebras. If the twin-group on such a trace is trivial, then the techniques in [5] can be applied. Otherwise, the twin-group on this trace is transitive. If the appropriate centrality is assumed, then this twin group is abelian. For example, the weak abelian property ensures that the twin group is abelian on every subset (Lemma 4.2). The calculations in this twin group can replace the calculations in the abelian groups of the modular case (see Theorem 4.3). This is the basis of the argument in Section 5.

Traces for prime quotients are important from another aspect, too: they reflect the global properties of the algebra. For example, if an algebra fails to be weakly abelian, then one can push this failure into such a trace. Then the previously described technique can be applied to transform such failures to even worse failures in the subpowers of the algebra. This explains another characteristic of this paper: almost all arguments and lemmas speak about the behavior of certain matrices within traces. Understanding these leads to the main results.

Most of the results are formulated for tolerances (compatible, reflexive and symmetric relations), and not congruences. We believe that tolerances are the natural objects when studying these topics. One of the reasons is that it is easier to factor tolerances than congruences, because we do not have to take transitive closures, which are hard to control. If T is a tolerance and δ is a congruence of $\mathbf{A}$, then $T/\delta = \{(a/\delta, b/\delta) : (a,b) \in T\}$ is a tolerance of $\mathbf{A}/\delta$, but it is not necessarily a congruence, not even if T itself is a congruence (unless $\delta \subseteq T$).

Even though we tried to describe the ideas informally, we cannot pretend that the paper can be read without some knowledge of tame congruence theory. The reader is referred to [3] and [7] for an introduction to the theory. The reader may want to brush up on E-minimal algebras, too, using these sources. We shall use some concepts and results of [5] as described above, so glancing at that paper might also be a good idea.

In the paper we use the usual terminology and notation. Boldface lower case letters usually denote vectors (sequences). For example, the notation $\mathbf{a}\ T\ \mathbf{b}$ means that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ for an appropriate n, and $(a_i, b_i) \in T$ for every i $(1 \le i \le n)$.

2. Preliminaries

In this section we summarize some basic facts about centrality and the twin relation that we shall use throughout the paper. All binary relations considered are assumed to be reflexive and symmetric. Let L and R be such relations of an algebra $\mathbf{A}$. By an L, R*-matrix* we mean a matrix of the form

$$\begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix},$$

where t is a polynomial of $\mathbf{A}$, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are vectors of $\mathbf{A}$ such that $\mathbf{a}\ L\ \mathbf{b}$ and $\mathbf{c}\ R\ \mathbf{d}$.

Definition 2.1 ([3], [4], [5])**.** Let L and R be reflexive and symmetric binary relations of an algebra $\mathbf{A}$, and δ a congruence of $\mathbf{A}$. We say that

(1) $\mathbf{C}(L, R; \delta)$ *holds* if for every L, R-matrix, the two elements in the first row are δ-related if and only if the two elements in the second row are δ-related;
(2) $\mathbf{W}(L, R; \delta)$ *holds* if for every L, R-matrix, if three of its elements are δ-related, then all four elements are δ-related;
(3) $\mathbf{R}(L, R; \delta)$ *holds* if for every L, R-matrix, if the two elements on the main diagonal are δ-related, then all four elements are δ-related;
(4) $\mathbf{S}(L, R; \delta)$ *holds* if and only if $\mathbf{C}(L, R; \delta)$ and $\mathbf{R}(L, R; \delta)$ both hold.

Extending the usual terminology of [3] from $\mathbf{C}$ to $\mathbf{W}$ and $\mathbf{S}$, we will express the fact that $\mathbf{W}(L, R; \delta)$ or $\mathbf{S}(L, R; \delta)$ holds by saying that L *weakly* or *strongly centralizes* R *modulo* δ. We will express $\mathbf{R}(L, R; \delta)$ by saying that L *rectangulates* R *modulo* δ.

The relation L is called *rectangular* if $\mathbf{R}(L, L; 0_{\mathbf{A}})$ holds, and $\mathbf{A}$ is called rectangular if $1_{\mathbf{A}} = A \times A$ is rectangular. Similarly, L is *weakly abelian,* if $\mathbf{W}(L, L; 0_{\mathbf{A}})$ holds.

Let R be a reflexive and symmetric binary relation of an algebra $\mathbf{A}$. Two polynomials $f(\mathbf{x})$ and $g(\mathbf{x})$ are called *R-twins*, if there exists a polynomial $t(\mathbf{x}, \mathbf{y})$ and vectors $\mathbf{u}\ R\ \mathbf{v}$ such that $f(\mathbf{x}) = t(\mathbf{x}, \mathbf{u})$ and $g(\mathbf{x}) = t(\mathbf{x}, \mathbf{v})$ holds for every $\mathbf{x}$. The polynomials occurring in the rows of L, R-matrices are R-twins, while the ones in the columns are L-twins. The twin relation plays an important role when considering polynomials of subdirect powers of an algebra. For example, if $\mathbf{T}$ is a tolerance of $\mathbf{A}^2$, then the unary polynomials of $\mathbf{T}$ are just pairs of T-twin unary polynomials of $\mathbf{A}$ acting componentwise.

Let E be a finite subset of $\mathbf{A}$. Consider the set S of all unary mappings of E to E that are restrictions of unary polynomials of $\mathbf{A}$. Then $\mathbf{S} = (S, \circ)$ is a semigroup, where $\circ$ denotes composition. The R-twin relation between unary polynomials is a reflexive and symmetric binary relation $\sim_R$ on S, which is obviously compatible with respect to composition. Let G be the set of permutations in S. Clearly, $\mathbf{G} = (G, \circ)$ is a group, we shall denote this group by $\mathbf{G}(E)$. We call E *permutational*, if all elements of S are either permutations of S, or are constant maps.

It is well-known, and easy to check, that every reflexive, compatible relation on a Maltsev algebra is a congruence. Hence, the relation $\sim_R$ restricted to $\mathbf{G}(E)$ is a congruence. The normal subgroup of $\mathbf{G}$ corresponding to this congruence is just the set of R-twins of the identity map. This subgroup is called the R-twin group on E, and is denoted by $\mathrm{Tw}(E, R)$.

In tame congruence theory the induced algebra $\mathbf{A}|_E$ is defined to have underlying set E, and its basic operations are the restrictions (to E) of those polynomials p of $\mathbf{A}$ under which E is closed, that is, for which $p(E, \ldots, E) \subseteq E$. Thus if E is permutational, then the unary part of the clone of $\mathbf{A}|_E$ is $\mathbf{G}(E)$. Therefore the congruences of $\mathbf{A}|_E$ are determined by this group. In the following lemma we describe the way this happens.

Recall that a group is said to act regularly on a set if it is transitive, and the stabilizer of each point is trivial. It is easy to see that an abelian group that acts transitively is always regular. The following lemma lists some well-known facts from the theory of permutation groups; its proof is left to the reader.

Lemma 2.2. *Let $\mathbf{A}$ be an algebra and E a finite permutational subset of A. Then the following hold.*

(1) *Every partition containing the one given by the orbits of $\mathbf{G}(E)$ is a congruence of $\mathbf{A}|_E$.*
(2) *Suppose that $\mathbf{G}(E)$ is transitive, let $b \in E$ be a fixed element, and H the stabilizer of b. Then the following hold.*
   (*i*) *The mapping $g(b) \mapsto gH$ is a one-to-one correspondence between E and the set of left cosets modulo H (on which $\mathbf{G}(E)$ acts via left multiplication).*

(*ii*) *The congruences of* $\mathbf{A}|_E$ *are in one-to-one correspondence with the subgroups of* $\mathbf{G}(N)$ *containing* H. *They are given by the left coset decompositions of* $\mathbf{G}(E)$ *modulo these subgroups, using the correspondence in* (*i*).

(*iii*) *If* N *is a normal subgroup of* $\mathbf{G}(E)$, *then its orbits on* E *give a congruence of* $\mathbf{A}|_E$, *which corresponds to the subgroup* HN *via* (*ii*).

Suppose that E is a finite E-trace of $\mathbf{A}$, that is, the intersection of the range of an idempotent polynomial of $\mathbf{A}$ and a class of a congruence of $\mathbf{A}$. Then by Lemma 2.4 of [3], every congruence of $\mathbf{A}|_E$ can be extended to $\mathbf{A}$. This holds, in particular, when $E = N$ is a trace for a minimal congruence of $\mathbf{A}$. In the latter case N is permutational, and the induced algebra $\mathbf{A}|_N$ is simple. Therefore we have the following corollary to the observations above.

Corollary 2.3. *Let* $\mathbf{A}$ *be a finite algebra and* N *a* $\langle 0_{\mathbf{A}}, \theta\rangle$*-trace for a minimal congruence* θ *of* $\mathbf{A}$. *Then* $\mathbf{G}(N)$ *is either transitive on* N, *or is trivial. The same holds for* $\mathrm{Tw}(N, R)$ *for every reflexive and symmetric binary relation* R.

The twin relation enables us to describe the induced algebras on certain subsets in special subdirect powers. For a reflexive and symmetric binary relation R and a finite or infinite cardinal κ, denote by $R^{[\kappa]}$ the κ-ary relation consisting of all tuples $(a_i : i < \kappa)$ such that $(a_i, a_j) \in R$ for every $i < j < \kappa$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^\kappa$ generated by $R^{[\kappa]}$ and the diagonal (that is, the set of constant sequences). Then the unary polynomials of $\mathbf{B}$ are sequences $(f_i : i < \kappa)$ of polynomials of $\mathbf{A}$ such that they are *simultaneous* R*-twins,* that is, they come from the same polynomial of $\mathbf{A}$, with the substitution of pairwise R-related tuples. In particular, these polynomials are pairwise R-twins. A very important observation is that conversely, under certain conditions, pairwise R-twin polynomials can be obtained from a single "father".

Lemma 2.4. *Let* E *be a finite subset of an algebra* $\mathbf{A}$, *and* R *a reflexive and symmetric binary relation of* $\mathbf{A}$. *Then any system of permutations of* E *that are pairwise* R*-twin polynomials is in fact a family of simultaneous* R*-twin polynomials of* E.

Proof. Let $\mathbf{G} = \mathbf{G}(E)$ with identity element e, and denote by K the set of all sequences of G^n that are simultaneous R-twins. Clearly, $\mathbf{K}$ is a subgroup of $\mathbf{G}^n$. Suppose that $(g_1, \ldots, g_n)$ is a sequence such that $g_i \sim_R g_j$ for every i, j. Thus, $g_i^{-1} g_j \in \mathrm{Tw}(E, R)$. Then obviously $(e, \ldots, e, g_1^{-1} g_i, e, \ldots, e) \in K$, and by multiplying these n elements we get that $(g_1^{-1} g_1, \ldots, g_1^{-1} g_n) \in K$. Finally, $(g_1, \ldots, g_1) \in K$, so $(g_1, \ldots, g_n) \in K$ and we are done. □

Thus if E is a finite permutational subset of $\mathbf{A}$, then this lemma can be used to describe the induced polynomial-permutations on the finite powers of E. To make sure that we have described all unary polynomials of E^k this way, we have to assume that E^k is also permutational, that is, that no polynomial permutation of E can be a twin of a constant map of E. Thus, for a subset E of $\mathbf{A}$ we say that $\mathbf{G}(E)$ is

$\sim_R$*-closed* if every R-twin of any permutation of E is also a permutation of E. This condition can be ensured by assuming an appropriate centrality relation.

Lemma 2.5. *Let E be a finite permutational E-trace of an algebra $\mathbf{A}$, and R a reflexive and symmetric binary relation of $\mathbf{A}$. Then $\mathbf{W}(R, E^2; 0_{\mathbf{A}})$ implies that $\mathbf{G}(E)$ is $\sim_R$-closed.*

Proof. Suppose that $f(x)$ and $g(x)$ are R-twin unary polynomials such that f is a permutation of E, but g is constant on E. By the finiteness of E, the inverse of f can be obtained in the form f^k for some k. Clearly, $f' = f^k f$ and $g' = f^k g$ are still R-twins, f' is the identity map on E, and g' is constant on E. Denote this constant by b and let $a \in E$, $a \neq b$. Then the R, N^2-matrix

$$\begin{bmatrix} f'(a) & f'(b) \\ g'(a) & g'(b) \end{bmatrix} = \begin{bmatrix} a & b \\ b & b \end{bmatrix}$$

gives a failure of $\mathbf{W}(R, E^2; 0_{\mathbf{A}})$, and so the statement is proved. □

We can now describe induced algebras on certain subsets in some subpowers. This description is the basis of several proofs in the paper. If T is a reflexive and symmetric binary relation on $\mathbf{A}$, then by T^κ we mean the binary relation on $\mathbf{A}^\kappa$, where two vectors are T^κ-related if and only if they are T-related componentwise.

Lemma 2.6. *Let $\mathbf{A}$ be an algebra, E a finite permutational E-trace of $\mathbf{A}$, and R a reflexive and symmetric binary relation of A such that $\mathbf{G}(E)$ is $\sim_R$-closed. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^\kappa$ generated by the diagonal and $R^{[\kappa]}$, and set $F = E^\kappa \cap B$. Then the following hold.*

1. *F is an E-trace of $\mathbf{B}$, and the induced algebra $\mathbf{B}|_F$ is permutational.*
2. *We have $\mathbf{G}(F) = (\sim_R)^{[\kappa]}|_F$. That is, the unary polynomials induced on F are the constant maps, and the sequences of pairwise R-twin unary polynomial-permutations of E acting componentwise on F.*
3. *Let T be a reflexive and binary relation on A. Then we have $\mathrm{Tw}(F, T^\kappa|_B) = \mathrm{Tw}(E, T)^\kappa \cap \mathbf{G}(F)$. That is, a sequence of polynomial-permutations of E is a T^k-twin of the identity map on F if and only if every component is a T-twin of the identity map on E.*

The analogous statements hold if we replace B and F by their respective subsets of almost constant vectors (that is, elements that are constant on a cofinite subset of κ). In this case we get the almost constant sequences of $(\sim_R)^{[\kappa]}$ in (2).

Proof. Most of the statements are easy calculations, so we only make some remarks. The assumption of $\mathbf{G}(E)$ being $\sim_R$-closed is used to show that every non-constant unary polynomial of F is a permutation. We explain how to use the finiteness of E to prove (2). Consider a system $(f_i : i < \kappa)$ of elements of $\mathbf{G}(E)$ that are pairwise R-twins. To show that these yield a unary polynomial of $\mathbf{B}|_F$ we have to make them simultaneous R-twins. Lemma 2.4 helps, because there are only finitely many different permutations on the finite set E. A similar argument works in (3): if all f_i are T-twins of the identity map of E, then they are simultaneous twins of the identity map by Lemma 2.4, because there are only finitely many different ones among them. Finally we point out that F contains the diagonal of E^κ, and therefore different sequences of polynomial-permutations of E will have different restrictions to F. $\square$

We now give a summary of the concepts of nilpotence that we shall need in this paper. Recall that a normal subgroup of a finite group G is nilpotent if and only of it centralizes each prime quotient of G. This concept has been extended to general algebras, and has been investigated in its most general form in [4] and [5]. Various forms of it turned out to be useful. In this paper we need only two of these concepts: the weakest and the strongest. The weakest one is defined as follows.

Definition 2.7. Let $\mathbf{A}$ be a finite algebra and T a tolerance of $\mathbf{A}$. We say that T is *barely nilpotent*, if T weakly centralizes every trace for every prime quotient of $\mathbf{A}$, that is, for every $\langle\delta,\theta\rangle$-trace N for every prime quotient $\langle\delta,\theta\rangle$ of $\mathbf{A}$ we have that $\mathbf{W}(T,N^2;\delta)$ holds. An algebra $\mathbf{A}$ is called barely nilpotent, if the tolerance $1_{\mathbf{A}}$ is barely nilpotent.

From the results in [4] it follows that *every homomorphic image of a weakly abelian algebra is barely nilpotent* (and satisfies the stronger forms of nilpotence coming from normal centrality as well). Since we shall prove that in a residually small variety, every barely nilpotent tolerance is weakly abelian, we shall get that in such varieties all concepts of nilpotence coming from ordinary or weak centrality coincide.

Strong nilpotence has been investigated in [5]. A tolerance T of a finite algebra is *strongly nilpotent* if it strongly centralizes every prime quotient of the algebra from both sides. This is equivalent to the much weaker condition that T rectangulates the traces of all prime quotients. We shall need the following observation from that paper, which shows that a trace weakly centralized by a tolerance T is rectangulated by this tolerance if and only if the T-twin group on this trace is trivial.

Lemma 2.8 ([5], Lemma 2.4). *Let N be a $\langle\delta,\theta\rangle$-trace for a prime quotient $\langle\delta,\theta\rangle$ of a finite algebra $\mathbf{A}$, and T a tolerance of $\mathbf{A}$. Then $\mathbf{R}(T,N^2;\delta)$ is equivalent to $\mathbf{W}(T,N^2;\delta)$ plus the condition that $\mathrm{Tw}(N,T)$ is trivial modulo δ, that is, every T-twin permutation of the identity map of N is equal to the identity map modulo δ.*

3. Residually small varieties

As explained in the Introduction, one of the main results of the paper is the following.

Theorem 3.1. *Let $\mathbf{A}$ be a finite algebra in a residually small variety. Then every barely nilpotent tolerance of $\mathbf{A}$ is weakly abelian. If $\mathbf{A}$ is barely nilpotent, then every member of $\mathsf{V}(\mathbf{A})$ is weakly abelian.*

The proof of this statement depends on the fact that in a residually small variety we can understand the behavior of T,T-matrices that are contained in traces weakly centralized by T. Using deeper methods, a complete description of such matrices is possible. However, to prove Theorem 3.1, the following lemma is sufficient.

Lemma 3.2. *Let* $\mathbf{A}$ *be a finite algebra in a residually small variety,* T *a tolerance of* $\mathbf{A}$*, and* N *a* $\langle 0_{\mathbf{A}}, \theta\rangle$*-trace for a minimal congruence* θ *of* $\mathbf{A}$ *such that* $\mathbf{W}(T, N^2; 0_{\mathbf{A}})$ *holds. Then there is no* T,T*-matrix*

$$\begin{bmatrix} v & u \\ v & v \end{bmatrix}$$

with $u \neq v$*,* $u, v \in N$*.*

We first show how this lemma implies Theorem 3.1. Let T be a barely nilpotent tolerance of $\mathbf{A}$. Suppose that T is not weakly abelian, then there exists a T,T-matrix

$$\begin{bmatrix} v & u \\ v & v \end{bmatrix},$$

where $u \neq v$. Let $\langle \delta, \theta\rangle$ be a prime quotient of $\mathbf{A}$ such that $(u, v) \in \theta - \delta$. By applying an appropriate unary polynomial we may assume that u and v are contained in a $\langle \delta, \theta\rangle$-trace N. As T is barely nilpotent, we have $\mathbf{W}(T, N^2; \delta)$. Replacing $\mathbf{A}$ by $\mathbf{A}/\delta$ and T by T/δ we may assume that $\delta = 0_{\mathbf{A}}$. Then Lemma 3.2 gives a contradiction, showing that T is weakly abelian.

Thus, if $\mathbf{A}$ itself is barely nilpotent, then it is weakly abelian. The weak abelian property is clearly preserved by direct products and subalgebras, hence each finite algebra $\mathbf{B} \in \mathsf{S}\,\mathsf{P}(\mathbf{A})$ is weakly abelian. Thus all homomorphic images of $\mathbf{B}$ are barely nilpotent (by the results of [4]), so they are weakly abelian by residual smallness. $\square$

The rest of this section is devoted to the proof of Lemma 3.2. The proof consists of three steps. First we apply a result of [5] to prove the case when $\mathrm{Tw}(N, T)$ is trivial. Corollary 2.3 shows that if $\mathrm{Tw}(N, T)$ is nontrivial, then it is transitive on N. In the second step we present a new construction of subdirectly irreducibles, which settles the case when $\mathrm{Tw}(N, T)$ is abelian and transitive on N. In the third step, we apply the result of the second step in an appropriate subpower of $\mathbf{A}$ to show that $\mathrm{Tw}(N, T)$ is always abelian.

For the first step let us quote the following construction of large subdirectly irreducibles from [5].

Lemma 3.3 ([5], Lemma 8.5)**.** *Let* N *be a* $\langle 0_{\mathbf{A}}, \theta\rangle$*-trace for a minimal congruence* θ *of a finite algebra* $\mathbf{A}$*, and* T *a tolerance of* $\mathbf{A}$ *such that* $\mathbf{R}(T, N^2; 0_{\mathbf{A}})$ *holds. Suppose that there exist*

(1) *pairwise* T*-related elements* $u, v, w \in N$ *which are not all equal, and*
(2) *pairs* $(a^\ell, b^\ell) \in T$ *for* $1 \le \ell \le m$ *such that the congruence of* $\mathbf{T}^{[3]}$ *generated by collapsing* (b^ℓ, a^ℓ, a^ℓ) *with* (a^ℓ, a^ℓ, b^ℓ) *for every* $1 \le \ell \le m$ *collapses* (v, v, v) *with* (u, v, w)*.*

Then $\mathsf{V}(\mathbf{A})$ *is residually large.*

This result immediately yields the case of Lemma 3.2 when the twin group on N is trivial. Indeed, assume that the conditions of this lemma hold, and $\mathrm{Tw}(N,T)$ is trivial. By Lemma 2.8 we get $\mathbf{R}(T, N^2; 0_{\mathbf{A}})$. Suppose that there is a T,T-matrix

$$\begin{bmatrix} v & u \\ v & v \end{bmatrix}$$

such that $u \neq v \in N$. This matrix can be written in the form

$$\begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix},$$

where t is a polynomial, $\mathbf{a}\ T\ \mathbf{b}$, and $\mathbf{c}\ T\ \mathbf{d}$. Apply Lemma 3.3 so that $w = v$ and the elements a^ℓ and b^ℓ are the components of $\mathbf{a}$ and $\mathbf{b}$, respectively. Define an m-ary polynomial on $\mathbf{T}^{[3]}$ by

$$s((x_1,y_1,z_1),\dots,(x_m,y_m,z_m)) = (t(\mathbf{x},\mathbf{d}), t(\mathbf{y},\mathbf{c}), t(\mathbf{z},\mathbf{c}))\,.$$

Then clearly

$$s((b^1,a^1,a^1),\dots,(b^m,a^m,a^m)) = (v,v,v)\,,$$

and

$$s((a^1,a^1,b^1),\dots,(a^m,a^m,b^m)) = (u,v,v)\,,$$

showing that (2) of Lemma 3.3 holds. Thus $\mathsf{V}(\mathbf{A})$ is residually large. This contradiction proves that such a T,T-matrix cannot exist.

For the second step of our proof we need a new construction of subdirectly irreducibles, whose origin is in the Freese-McKenzie theorem ([1], Theorem 10.15). We need to formulate this result more generally, in order to be able to complete the third step. Therefore, it is formulated for permutational E-traces rather than for traces of minimal quotients. (We could have easily generalized the previous lemma for permutational E-traces, too.)

Lemma 3.4. *Let N be a permutational E-trace of a finite algebra $\mathbf{A}$, and T a tolerance of $\mathbf{A}$ such that N is $\sim_T$-closed, and $\mathrm{Tw}(N,T)$ is abelian and transitive on N. Suppose that there exist*

1. *T-related elements $u, v \in N$ which are not equal, and*
2. *pairs $(a^\ell, b^\ell) \in T$ for $1 \le \ell \le m$ such that the congruence of $\mathbf{T}^{[3]}$ generated by collapsing (b^ℓ, a^ℓ, a^ℓ) with (a^ℓ, a^ℓ, b^ℓ) for every $1 \le \ell \le m$ collapses (v,v,v) with (u,v,v).*

Then $\mathsf{V}(\mathbf{A})$ is residually large.

Proof. First notice that $N^2 \subseteq T$. Indeed, let $a, b \in N$. Then there exists a g in $\mathbf{H} = \mathrm{Tw}(N,T)$ such that $g(a) = b$. Since g is a T-twin of the identity, we get that $g(a)$ and a are T-related. Hence $N^2 \subseteq T$ indeed. Note also that the group $\mathbf{H}$ acts regularly on N, since it is abelian and transitive.

Let κ be an infinite cardinal, and $\mathbf{B}$ the algebra of all almost constant elements in $T^{[\kappa]}$. Let $M = N^\kappa \cap B$. Lemma 2.6 shows that M is a permutational E-trace of $\mathbf{B}$, and describes the group $\mathbf{G}(M)$ to be the set of all almost constant sequences $(ff_i : i < \kappa)$ where $f \in \mathbf{G}(E)$ and $f_i \in H$.

Consider the subgroup $\mathbf{K}$ of $\mathbf{H}^\kappa$ consisting of those elements whose coordinates are equal to the identity element $e \in \mathbf{G}(E)$ with at most finitely many exceptions, and the product of these exceptional elements is e. The elements of K act on M componentwise. As $\mathbf{H}$ is abelian and normal in $\mathbf{G}(E)$, from the description of $\mathbf{G}(M)$ we see that $\mathbf{K}$ is normal in $\mathbf{G}(M)$. Hence its orbits form a congruence of $\mathbf{B}|_M$ by Lemma 2.2. Since M is an E-trace, this congruence can be extended to a congruence ψ of $\mathbf{B}$ (see Lemma 2.4 in [3]).

Let $\mathbf{v}$ be the element of M which is constant v, and denote by $\mathbf{u}^i$ the element of B whose every component is v except that the i-th component is u. Note that $\mathbf{v}$ and $\mathbf{u}^i$ are not ψ-related. Indeed, suppose that for some $\mathbf{f} = (f_i : i < \kappa) \in K$ we have $\mathbf{f}(\mathbf{v}) = \mathbf{u}^i$. By the regularity of H we see that $f_j = e$ with the only exception of $j = i$, but $f_i \neq e$, because $u \neq v$. Therefore the product of all f_j's is not the identity element, which is a contradiction. Notice also that all elements $\mathbf{u}^i$ are in the same ψ-class. Indeed, there is a $h \in H$ mapping v to u, and so we can take $\mathbf{u}^i$ to $\mathbf{u}^j$ by the system of permutations whose components are all e except that the i-th component is h^{-1} and the j-th component is h. These permutations multiply to e, and so we moved $\mathbf{u}^i$ to $\mathbf{u}^j$ by an element of K.

Choose a congruence $\psi_0 \geq \psi$ of $\mathbf{B}$ that is maximal among the ones separating $\mathbf{v}$ and $\mathbf{u}^0$. Then $\mathbf{B}/\psi_0$ is a a subdirectly irreducible factor of $\mathbf{B}$, let λ denote its cardinality. To finish the proof it is sufficient to show that $\lambda \geq \kappa$.

Denote by ${}^i\mathbf{d}^\ell$ the element of B whose every component is a^ℓ except that the i-th component is b^ℓ. We show that for each $i \neq j < \kappa$ there exists an $1 \leq \ell \leq m$ such that $({}^i\mathbf{d}^\ell, {}^j\mathbf{d}^\ell) \notin \psi_0$.

Indeed, suppose that this fails for some $i \neq j$. Let $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ consisting of those functions that are constant on the set $\kappa - \{i, j\}$. For $\mathbf{c} \in C$ let $\varphi(\mathbf{c}) = (c_i, c_k, c_j)$, where $k \in \kappa - \{i, j\}$ is arbitrary. Clearly, $\varphi : \mathbf{C} \to \mathbf{T}^{[3]}$ is an isomorphism. We have $\varphi({}^i\mathbf{d}^\ell) = (b^\ell, a^\ell, a^\ell)$, $\varphi({}^j\mathbf{d}^\ell) = (a^\ell, a^\ell, b^\ell)$, $\varphi(\mathbf{v}) = (v, v, v)$, and $\varphi(\mathbf{u}^i) = (u, v, v)$. Let $\theta = \varphi(\psi_0|_C)$, this is a congruence of $\mathbf{T}^{[3]}$. It collapses every (b^ℓ, a^ℓ, a^ℓ) with (a^ℓ, a^ℓ, b^ℓ), so condition (2) of the lemma implies that $\mathbf{v}\ \psi_0\ \mathbf{u}^i$. But $\mathbf{u}^i\ \psi\ \mathbf{u}^0$ by our remark above, and so we have that $\mathbf{v}\ \psi_0\ \mathbf{u}^0$, which is a contradiction, proving the statement of the previous paragraph.

Now define a mapping $g : \kappa \to (B/\psi_0)^m$ by $g(i) = ({}^i\mathbf{d}^1/\psi_0, \ldots, {}^i\mathbf{d}^m/\psi_0)$. What we have just proved means exactly that g is injective. Therefore $\kappa \leq \lambda^m = \lambda$, proving the statement of the lemma. $\square$

Specializing the statement of this lemma exactly as in the case of Lemma 3.3 we get the case of Lemma 3.2, when $\mathrm{Tw}(N, T)$ is abelian and transitive.

Corollary 3.5. *Let* $\mathbf{A}$ *be a finite algebra in a residually small variety,* T *a tolerance of* $\mathbf{A}$*, and* N *a permutational* E*-trace of* $\mathbf{A}$ *such that* N *is* $\sim_T$*-closed. Suppose that* $\mathrm{Tw}(N,T)$ *is abelian and acts transitively on* N*. Then there is no* T,T*-matrix of the form*

$$\begin{bmatrix} v & u \\ v & v \end{bmatrix}$$

with $u \neq v$*,* $u, v \in N$*.*

Thus (taking Lemma 2.5 into consideration) we have completed the second step of the proof. Hence to complete the third step it is sufficient to prove that $\mathrm{Tw}(N,T)$ is always abelian. This has first been proved using a group-theoretic lemma based on Burnside's theorem on normal complements, which is found in [8], Theorem 0. Later, a much simpler proof of the same group-theoretic lemma has been given by Gyula Lakos. Examining his proof, we came up with the following elementary argument.

Lemma 3.6. *Let* $\mathbf{A}$ *be a finite algebra in a residually small variety,* T *a tolerance of* $\mathbf{A}$*, and* N *a permutational* E*-trace of* $\mathbf{A}$ *such that* $\mathbf{W}(T, N^2; 0_{\mathbf{A}})$ *holds. Then* $\mathrm{Tw}(N,T)$ *is abelian.*

Proof. The induced algebra $\mathbf{A}|_N$ is permutational, so it is a minimal algebra, and hence Pálfy's Theorem applies. If its type is nonabelian, then $|N| = 2$, and so the twin group must be a subgroup of the cyclic group of order two, and we are done. (In fact, in this case the twin group must be trivial because $\mathbf{W}(T, N^2; 0_{\mathbf{A}})$ implies that T must not contain a nontrivial pair from N).

If the type of $\mathbf{A}|_N$ is $\mathbf{2}$, then N has an abelian group structure. Suppose that $f(x) \in \mathbf{G}(N)$ is a T-twin of the identity map $e(x)$ of N. Then $f(x) - e(x)$ and $e(x) - e(x)$ are also T-twin unary polynomials. The second one is constant, so by $\mathbf{W}(T, N^2; 0_{\mathbf{A}})$ we see that $f(x) - e(x)$ is constant, too. Therefore f is a translation. The group of translations is abelian, and so we are done in this case, too.

Assume now that the type of $\mathbf{A}|_N$ is $\mathbf{1}$. The idea is to go to a power, find an E-trace whose twin group is already abelian, and then apply the (already proved) Corollary 3.5. Let $g_1, g_2 \in \mathbf{H} = \mathrm{Tw}(N,T)$, and $h = g_1^{-1} g_2^{-1} g_1 g_2$. We have to show that h is the identity map. Let $|N| = k$, and $\mathbf{B}$ be the subalgebra of $\mathbf{A}^k$ generated by the diagonal and the set N^k. As $\mathbf{A}|_N$ is essentially unary, the N^2-twin relation is trivial on N. Hence applying Lemma 2.6 to the relation $R = N^2$ we see that N^k is a permutational E-trace of $\mathbf{B}$, and $\mathbf{G}(N^k)$ consists of all constant sequences $\hat{f} = (f, \ldots, f)$, where $f \in \mathbf{G}(N)$.

The equivalence relation consisting of the orbits of $\mathbf{G}(N^k)$ is a congruence of the induced algebra on N^κ, and so each of these orbits is an E-trace. Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a listing of the elements of N, and M the orbit containing $\mathbf{n}$. Then $\mathbf{G}(M)$ is the set of restrictions of the elements of $\mathbf{G}(N^k)$ to M. However, the stabilizer of $\mathbf{n}$ in $\mathbf{G}(M)$ is already trivial: if $\hat{f}$ fixes $\mathbf{n}$, then $f(n_i) = n_i$ for each i, and so f is the identity map.

Since $\mathbf{G}(M)$ is transitive on M, it is in fact regular. Thus the congruences of $\mathbf{B}|_M$ correspond to the subgroups of $\mathbf{G}(M)$ in the usual manner (see Lemma 2.2).

Denote by $\mathbf{C}$ the cyclic subgroup of $\mathbf{G}(M)$ generated by the permutation $\hat{h}$, and let K be the $\mathbf{C}$-orbit containing $\mathbf{n}$. Since $\mathbf{G}(M)$ acts regularly on M, the orbits of $\mathbf{C}$ form a congruence of $\mathbf{B}|_M$. Thus K is also an E-trace. The group $\mathbf{G}(K)$ consists of the restrictions of all elements of $\mathbf{G}(M)$ to K under which K is closed. But if $\hat{f}$ is such, then $\hat{f}(\mathbf{n}) = \hat{h}^m(\mathbf{n})$ for some m, and therefore $f = h^m$. Thus K is a permutational E-trace, and $\mathbf{G}(K)$ is the set of all powers of $\hat{h}$. Thus $\mathbf{G}(K) \cong \mathbf{C}$ is a cyclic group, and so the twin-group for any relation must be abelian, too.

Now consider the relation T^k restricted to $\mathbf{B}$. From Lemma 2.6 (3) we see that $\mathrm{Tw}(K, T^k|_B) = \mathbf{G}(K)$, since $h \in \mathbf{H} = \mathrm{Tw}(N, T)$. Therefore this twin group is transitive as well. It is also clear that $\mathbf{G}(K)$ is closed under the T^k-twin relation. Thus the initial conditions of Corollary 3.5 are satisfied for K and T^k. We shall set up a T^k, T^k-matrix in K.

Let e denote the identity map of N. We know that $g_1, g_2 \in \mathbf{H}$, so their inverses are in $\mathbf{H}$ as well. Thus we have polynomials and T-related parameter sequences such that the following equalities hold for every $n \in N$.

$$\begin{array}{rclcrcl}
g_1(n) & = & t_1(n, \mathbf{u}^1) & \qquad & g_2(n) & = & t_2(n, \mathbf{u}^2) \\
e(n) & = & t_1(n, \mathbf{v}^1) & & e(n) & = & t_2(n, \mathbf{v}^2) \\
g_1^{-1}(n) & = & s_1(n, \mathbf{w}^1) & & g_2^{-1}(n) & = & s_2(n, \mathbf{w}^2) \\
e(n) & = & s_1(n, \mathbf{z}^1) & & e(n) & = & s_2(n, \mathbf{z}^2) \,.
\end{array}$$

We could actually assume that $t_1 = t_2 = s_1 = s_2$, but this would obscure the idea of the proof. We are interested in the expression $g_1^{-1} g_2^{-1} g_1 g_2(n)$. This can be written as

$$s_1(s_2(t_1(t_2(n, \mathbf{u}^2), \mathbf{u}^1), \mathbf{w}^2), \mathbf{w}^1) \,.$$

Therefore we see that

$$\begin{bmatrix} g_1^{-1} e^{-1} g_1 e(n) & g_1^{-1} g_2^{-1} g_1 g_2(n) \\ e^{-1} e^{-1} e e(n) & e^{-1} g_2^{-1} e g_2(n) \end{bmatrix}$$

is a T, T-matrix in N for every fixed value of n. If we change the value of n within N, then we get N-twin polynomials of the entries of this matrix. Hence

$$\begin{bmatrix} \hat{g}_1^{-1} \hat{e}^{-1} \hat{g}_1 \hat{e}(\mathbf{n}) & \hat{g}_1^{-1} \hat{g}_2^{-1} \hat{g}_1 \hat{g}_2(\mathbf{n}) \\ \hat{e}^{-1} \hat{e}^{-1} \hat{e} \hat{e}(\mathbf{n}) & \hat{e}^{-1} \hat{g}_2^{-1} \hat{e} \hat{g}_2(\mathbf{n}) \end{bmatrix}$$

is a T^k, T^k-matrix in $\mathbf{B}$. By evaluating the compositions we see that this matrix is equal to

$$\begin{bmatrix} \mathbf{n} & \hat{h}(\mathbf{n}) \\ \mathbf{n} & \mathbf{n} \end{bmatrix} .$$

Corollary 3.5 implies that $\hat{h}(\mathbf{n}) = \mathbf{n}$, so $h = e$, and the proof is complete. $\square$

4. Weakly abelian varieties

In this section we shall investigate weakly abelian varieties. We start by presenting a useful way to construct T,T-matrices using twin polynomials.

Lemma 4.1. *Let T be a tolerance of an algebra $\mathbf{A}$, and $E \subseteq A$. Suppose that*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a T,T-matrix with $a,b,c,d \in E$, and $g \in \mathrm{Tw}(E,T)$. Then

$$\begin{bmatrix} g(a) & b \\ g(c) & d \end{bmatrix}$$

is a T,T-matrix, too. Similarly, we can prefix the other column, or any row with g, or the entire matrix with any unary polynomial, and shall still get a T,T-matrix.

Proof. Suppose that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix}.$$

As $g \in \mathrm{Tw}(E,T)$, we can write $g(x) = s(x,\mathbf{u})$ for some polynomial s such that $s(x,\mathbf{v})$ is the identity map on E for some vector $\mathbf{v}$ that is T-related to $\mathbf{u}$. Hence

$$\begin{bmatrix} g(a) & b \\ g(c) & d \end{bmatrix} = \begin{bmatrix} s(t(\mathbf{a},\mathbf{c}),\mathbf{u}) & s(t(\mathbf{a},\mathbf{d}),\mathbf{v}) \\ s(t(\mathbf{b},\mathbf{c}),\mathbf{u}) & s(t(\mathbf{b},\mathbf{d}),\mathbf{v}) \end{bmatrix}$$

is clearly a T,T-matrix (with respect to the composition of s and t). $\square$

Our first observation is to show that in a weakly abelian variety, all twin-groups are abelian.

Lemma 4.2. *Let $\mathbf{A}$ be a finite algebra and T a weakly abelian tolerance of $\mathbf{A}$. Then for every subset E of $\mathbf{A}$, $\mathrm{Tw}(E,T)$ is an abelian group.*

Proof. Let $f,g \in \mathrm{Tw}(E,T)$ and $a \in E$. As E is finite, f^{-1} and g^{-1} are also twin polynomials of the identity map on E. Use the previous lemma four times: we start with the trivial T,T matrix whose all four entries are a, and apply g to the first row, then f to the first column, then g^{-1} to the first row, and finally f^{-1} to the first column. The resulting T,T matrix is

$$\begin{bmatrix} f^{-1}g^{-1}fg(a) & g^{-1}g(a) \\ f^{-1}f(a) & a \end{bmatrix} = \begin{bmatrix} f^{-1}g^{-1}fg(a) & a \\ a & a \end{bmatrix}.$$

As T is weakly abelian, we get that $f^{-1}g^{-1}fg(a) = a$. Thus $fg(a) = gf(a)$. $\square$

The equivalence classes for Abelian congruences on an algebra in a congruence modular variety can be considered as modules: the ring is given by the action of the algebra on this class via unary polynomials. We shall now prove a generalization of this statement. We show that if the twin group for a weakly abelian tolerance is regular on a subset, then that subset can be considered a module.

Theorem 4.3. *Let E be a subset of a finite algebra $\mathbf{A}$, and T a weakly abelian tolerance of $\mathbf{A}$. Suppose that the group $\mathbf{H} = \mathrm{Tw}(E, T)$ is transitive on E. Fix $0 \in E$ arbitrarily.*

1. *For every $w \in E$ there is a unique element h_w of $\mathbf{H}$ such that $h_w(0) = w$. Identify w with h_w. This defines an abelian group on E such that 0 is the zero element. We have $h_u(v) = u + v$ for every $u, v \in E$.*
2. *If p is a unary polynomial of $\mathbf{A}|_E$, then p can be written as $p(x) = \lambda(x) + c$ ($x \in E$), where $c \in E$, and λ is a unary polynomial, which is an endomorphism of the group on E defined in (1).*
3. *Let S be the set of unary polynomials of $\mathbf{A}|_E$ that fix 0. Then the closure of S under addition and subtraction in the endomorphism ring of $(E, +)$ will be a subring $\mathbf{R}$, and E becomes an $\mathbf{R}$-module. Every unary polynomial of $\mathbf{A}|_E$ is a unary polynomial of this module. The congruences of this module and of $\mathbf{A}|_E$ are the same.*
4. *The T, T-matrices in E are exactly the matrices*
$$\begin{bmatrix} u & v \\ w & z \end{bmatrix}$$
satisfying $u + z = w + v$.

Proof. The group $\mathbf{H} = \mathrm{Tw}(E, T)$ is abelian by the previous lemma, and as it is transitive, it is regular, proving (1).

Next we prove (4). Suppose that the matrix in its statement satisfies $u + z = w + v$. Then
$$\begin{bmatrix} u & v \\ w & z \end{bmatrix} = \begin{bmatrix} h_w h_w^{-1} h_u(0) & h_w h_w^{-1} h_v(0) \\ h_w h_u^{-1} h_u(0) & h_w h_u^{-1} h_v(0) \end{bmatrix},$$
which is indeed a T, T-matrix. Conversely, if we have a T, T-matrix
$$\begin{bmatrix} u & v \\ w & z \end{bmatrix},$$
then create the new T, T-matrix
$$\begin{bmatrix} h_w h_u^{-1}(u) & h_w h_v^{-1}(v) \\ h_u h_u^{-1}(w) & h_u h_v^{-1}(z) \end{bmatrix} = \begin{bmatrix} w & w \\ w & u - v + z \end{bmatrix}.$$
From the fact the T is weakly abelian we get that $u - v + z = w$. Thus (4) is proved.

Now let p be a unary polynomial on $\mathbf{A}|_E$, and $u, v \in E$. Then
$$\begin{bmatrix} p h_u h_v(0) & p h_0 h_v(0) \\ p h_u h_0(0) & p h_0 h_0(0) \end{bmatrix}$$

is a T, T-matrix by Lemma 4.1. Calculating its elements and applying (4) we get that $p(u + v) + p(0) = p(u) + p(v)$. Thus $\lambda(x) = p(x) - p(0)$ and $c = p(0)$ satisfy the conditions in (2). The function λ is a unary polynomial, because it is equal to $h_{-p(0)}p(x)$.

Finally we prove (3). Since the set S is closed under composition, $\mathbf{R}$ is indeed a ring. If $\lambda \in S$, then $\lambda(x) + c = h_c\lambda(x)$ is a unary polynomial of $\mathbf{A}|_E$. Therefore the only nontrivial statement left to be proved is that every congruence θ of $\mathbf{A}|_E$ is a module-congruence. Since the translations h_w of the group structure on E are unary polynomials of $\mathbf{A}|_E$, the congruence θ is a group-congruence. Thus the class containing 0 is a subgroup, which is closed under the elements of S. Hence it is closed under finite sums and differences of the elements of S, and therefore it is an $\mathbf{R}$-submodule. $\square$

The following lemma is our main tool to understand algebras in a weakly abelian variety.

Lemma 4.4. *Let $\mathbf{A}$ be a finite algebra such that $\mathsf{HS}(\mathbf{A}^3)$ is weakly abelian, θ a minimal congruence of $\mathbf{A}$, and T a tolerance of $\mathbf{A}$ such that $\mathbf{R}(T, \theta; 0_{\mathbf{A}})$ holds. Then for every T, T-matrix*

$$\begin{bmatrix} u & v \\ w & u \end{bmatrix}$$

such that $u \;\theta\; v$ we have that $u = v$.

Proof. Let

$$\begin{bmatrix} u & v \\ w & u \end{bmatrix} = \begin{bmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{bmatrix},$$

where t is a polynomial, $\mathbf{a} \; T \; \mathbf{b}$, $\mathbf{c} \; T \; \mathbf{d}$, and suppose that $(u, v) \in \theta - 0_{\mathbf{A}}$. By composing t with an appropriate unary polynomial we may assume that u and v are different elements of a θ-trace N.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}^3$ with underlying set $T^{[3]}$. We define some vectors in the algebra $\mathbf{B}$. Let

$$\begin{aligned} \mathbf{a}' &= (\mathbf{a},\mathbf{a},\mathbf{b}) \\ \mathbf{b}' &= (\mathbf{a},\mathbf{a},\mathbf{a}) \\ \mathbf{c}' &= (\mathbf{c},\mathbf{c},\mathbf{d}) \\ \mathbf{d}' &= (\mathbf{c},\mathbf{d},\mathbf{d}). \end{aligned}$$

By this notation we mean that, say, $\mathbf{a}'$ has the same length as $\mathbf{a}$, and the i-th component of $\mathbf{a}'$ is (a_i, a_i, b_i), where a_i is the i-th component of $\mathbf{a}$ and b_i is the i-th component of $\mathbf{b}$. Then

$$\begin{bmatrix} t(\mathbf{a}',\mathbf{c}') & t(\mathbf{a}',\mathbf{d}') \\ t(\mathbf{b}',\mathbf{c}') & t(\mathbf{b}',\mathbf{d}') \end{bmatrix} = \begin{bmatrix} (u,u,u) & (u,v,u) \\ (u,u,v) & (u,v,v) \end{bmatrix}$$

is a T^3, T^3-matrix in $\mathbf{B}$. We shall construct a congruence ρ of $\mathbf{B}$ that collapses three elements of this matrix, but not the fourth one. This will contradict the fact that $\mathbf{B}/\rho$ is weakly abelian, and will therefore finish the proof.

Let $M = N^3 \cap B$. Lemma 2.6 implies that M is a permutational E-trace of $\mathbf{B}$, and $\mathbf{G}(M)$ consists of triples of pairwise T-twin permutations of $\mathbf{G}(N)$. From $\mathbf{R}(T, \theta; 0_{\mathbf{A}})$ we get that the three permutations in the triples are equal. Therefore $\mathbf{G}(M) = \{(f, f, f)|_M : f \in \mathbf{G}(N)\}$.

Consider the partition of M defined by the following rule. Triplets having three different entries form singleton classes, and for every $x \in N$, triplets containing the element x at least twice form a class. This is obviously a $\mathbf{G}(M)$-invariant partition of M, and therefore it is a congruence of $\mathbf{B}|_M$. Since M is an E-trace, this partition can be extended to a congruence ρ of $\mathbf{B}$. By the definition of ρ we see that the sets

$$\{(u, u, u), (u, u, v), (u, v, u)\} \quad \text{and} \quad \{(u, v, v)\}$$

are contained in two different ρ-classes. □

Corollary 4.5. *In a weakly abelian variety, every strongly nilpotent tolerance of every finite algebra is rectangular.*

Proof. Let $\mathbf{A}$ be a finite algebra in a weakly abelian variety, and T a strongly nilpotent tolerance of $\mathbf{A}$. Suppose that T is not rectangular. Then there exists a T, T-matrix

$$\begin{bmatrix} u & v \\ w & u \end{bmatrix}$$

such that $u \neq v$. Let δ be a maximal congruence of $\mathbf{A}$ not containing the pair (u, v), and θ its unique cover. The fact that T is strongly nilpotent implies that $\mathbf{R}(T, \theta; \delta)$ holds. Hence we can apply Lemma 4.4 to the algebra $\mathbf{A}/\delta$, the congruence θ/δ and tolerance T/δ to get a contradiction. □

Corollary 4.6. *A finite strongly nilpotent algebra generates a residually small variety if and only if this variety is weakly abelian (if and only if this variety is rectangular). If so, the variety has a finite residual bound.*

Proof. It is shown in [5] that a finite strongly nilpotent algebra generates a residually small variety iff this variety is rectangular, and if so, it has a finite residual bound. Such varieties are of course weakly abelian. It is also shown in [5] that all finite members of a variety generated by a finite, strongly nilpotent algebra are also strongly nilpotent. So if such a variety is weakly abelian, then by Corollary 4.5, all finite algebras, and hence all algebras in this variety are rectangular. □

Now we turn to the investigation of E-minimal algebras. Recall that a finite algebra is called E-minimal if it is minimal with respect to all of its prime quotients. Equivalently, a finite algebra $\mathbf{A}$ is E-minimal if and only if every unary polynomial p

of $\mathbf{A}$ is either a permutation of A, or collapses all prime congruence quotients of $\mathbf{A}$. This latter statement means that if $\delta \prec \theta$ are congruences of $\mathbf{A}$, then $p(\theta) \subseteq \delta$.

As shown in [3], every E-minimal algebra is either a two-element algebra, or its typeset is **2**, or its typeset is **1**. These solvable E-minimal algebras are characterized in [3], Theorem 13.7, and in [7], Theorem 4.4, respectively. From these characterizations, and from the results in [4] it follows that E-minimal algebras of types **2** and **1** are barely nilpotent. We also mention that E-minimal algebras of type **2** are Maltsev. A Maltsev algebra is weakly abelian if and only if it is abelian, and in this case the variety it generates has a finite residual bound. However, the arguments below work for these types of E-minimal algebras, too, there is no need to make a distinction.

Lemma 4.7. *Let $\mathbf{A}$ be an E-minimal algebra such that $\mathsf{H}\,\mathsf{S}(\mathbf{A}^3)$ is weakly abelian, and T a tolerance of $\mathbf{A}$. Let ρ denote the partition given by the orbits of $\mathrm{Tw}(A,T)$. Then for every T,T-matrix*

$$\begin{bmatrix} u & v \\ w & u \end{bmatrix}$$

we have that $(u,v) \in \rho$.

Proof. Suppose that this statement fails, and choose a failure so that $\theta = \mathrm{Cg}(u,v)$ be as small as possible. We show that $\theta \cap \rho$ is a congruence of $\mathbf{A}$.

Indeed, let $(a,b) \in \theta \cap \rho$, and p a unary polynomial of $\mathbf{A}$. Since p preserves θ, we have to prove that $p(a)$ and $p(b)$ are ρ-related. As $(a,b) \in \rho$ there exists a $g \in \mathrm{Tw}(A,T)$ such that $g(a) = b$. Suppose first that p is a permutation of $\mathbf{A}$. Then pgp^{-1} carries $p(a)$ to $p(b)$, and it is still a T-twin of the identity map pp^{-1} of A. Therefore $(p(a),p(b)) \in \rho$. Now suppose that p is not a permutation of A. By E-minimality, p is collapsing for every prime quotient of $\mathbf{A}$. Thus $(p(a),p(b))$ is contained in every lower cover of θ. Therefore $\theta = \mathrm{Cg}(u,v) > \mathrm{Cg}(p(a),p(b))$. Applying Lemma 4.1 to the trivial T,T-matrix with all entries equal to a we get that

$$\begin{bmatrix} pg(a) & p(a) \\ pg^2(a) & pg(a) \end{bmatrix}$$

is a T,T-matrix. The elements in the top row are $pg(a) = p(b)$ and $p(a)$, and we have seen that this pair generates a smaller congruence than $\theta = \mathrm{Cg}(u,v)$. By the minimality assumption on θ we get that $(p(a),p(b)) \in \rho$. Hence we have proved that $\theta \cap \rho$ is indeed a congruence.

The pair (u,v) shows that $\theta \cap \rho < \theta$. Let $\delta \geq \theta \cap \rho$ be a lower cover of θ. We show that $\mathbf{R}(T,\theta;\delta)$ holds. We have $\mathbf{W}(T,\theta;\delta)$, since $\mathsf{H}(\mathbf{A})$ is weakly abelian. Thus by Lemma 2.8 it is sufficient to show that the T/δ-twin group on N/δ is trivial for every $\langle \delta,\theta\rangle$-trace N. In other words, we have to show that if f and g are T-twin unary polynomials of $\mathbf{A}$ mapping N into N, f is the identity map on N modulo δ, and g is a permutation of N modulo δ, then they agree on N modulo δ. Since these unary polynomials do not collapse θ to δ, they are both permutations of $\mathbf{A}$ by E-minimality. Therefore $f^{-1}g$ is a T-twin of the identity map of $\mathbf{A}$, which still maps N to N. Thus for every element $a \in N$, the elements $f^{-1}g(a)$ and a are ρ-related. They are θ-related, too, because $N^2 \subseteq \theta$, and from $\theta \cap \rho \leq \delta$ we get that $f^{-1}g(a)\ \delta\ a$. Hence $f^{-1}g$ induces the identity map on N/δ. Thus $\mathbf{R}(T, \theta; \delta)$ holds indeed.

Now Lemma 4.4 applied to $\mathbf{A}/\delta$, T/δ and θ/δ shows that $(u, v) \in \delta$. This contradiction to $\delta < \theta = \mathrm{Cg}(u, v)$ proves the lemma. $\square$

Corollary 4.8. *Let* $\mathbf{A}$ *be an* E*-minimal algebra such that* $\mathsf{H}\,\mathsf{S}(\mathbf{A}^3)$ *is weakly abelian, and* T *a tolerance of* $\mathbf{A}$*. Then the orbits of* $\mathrm{Tw}(A, T)$ *form a congruence relation* ρ *of* $\mathbf{A}$*, and* $\mathbf{R}(T, T; \rho)$ *holds.*

Proof. To show that ρ is a congruence, let $(a, b) \in \rho$, and p a unary polynomial of $\mathbf{A}$. We have to prove that $(p(a), p(b)) \in \rho$. From $(a, b) \in \rho$ we get that there is a $g \in \mathrm{Tw}(A, T)$ such that $g(a) = b$. Look at the T, T-matrix

$$\begin{bmatrix} pg(a) & p(a) \\ pg^2(a) & pg(a) \end{bmatrix}.$$

Lemma 4.7 implies that $(pg(a), p(a)) = (p(b), p(a)) \in \rho$.

To prove that $\mathbf{R}(T, T; \rho)$ holds suppose that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a T, T-matrix such that $(a, d) \in \rho$. We have to prove that all four elements a, b, c, d are ρ-related. Let $g \in \mathrm{Tw}(A, T)$ be such that $g(d) = a$. Look at the T, T-matrix

$$\begin{bmatrix} a & b \\ g(c) & g(d) \end{bmatrix}.$$

Lemma 4.7 yields that $(a, b) \in \rho$. By symmetry we see that $(a, c) \in \rho$ as well. $\square$

Corollary 4.9. *Let* $\mathbf{A}$ *be an* E*-minimal algebra such that* $\mathsf{V}(\mathbf{A})$ *is weakly abelian. Then the orbits of* $\mathrm{Tw}(A, 1_{\mathbf{A}})$ *form a congruence relation* ρ *of* $\mathbf{A}$*, and* A/ρ *is essentially unary.*

To prove this corollary we take $T = 1_{\mathbf{A}}$ in Corollary 4.8. Then we get that A/ρ is rectangular (and still E-minimal). To finish the proof, we need the following result.

Theorem 4.10. *A rectangular* E*-minimal algebra generates a weakly abelian variety if and only if it is essentially unary.*

Proof. Let $\mathbf{A}$ be a rectangular E-minimal algebra generating a weakly abelian variety. Then this variety is rectangular by Corollary 4.6. In [5], Theorem 7.7 an equational characterization of finitely generated rectangular varieties is given. This theorem is too complicated to quote here in its general form. It implies that if $\mathbf{A}$ fails to be essentially unary, then $\mathbf{A}$ has an at least binary term $m(\mathbf{x}, \mathbf{z})$ that depends on all of its variables, and another term $k(\mathbf{x}, y)$ such that the identity $k(\mathbf{x}, m(\mathbf{y}, \mathbf{z})) = m(\mathbf{x}, \mathbf{z})$

holds in $\mathbf{A}$. As m depends on $\mathbf{z}$, there exist vectors $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{c}'$ in A such that $u = m(\mathbf{a}, \mathbf{c}) \neq m(\mathbf{a}, \mathbf{c}') = u'$. Hence,

$$k(\mathbf{a}, m(\mathbf{a}, \mathbf{c})) = m(\mathbf{a}, \mathbf{c}) \neq m(\mathbf{a}, \mathbf{c}') = k(\mathbf{a}, m(\mathbf{a}, \mathbf{c}')) .$$

Thus the unary polynomial $p(y) = k(\mathbf{a}, y)$ has two different fixed points u and u'. This polynomial has an idempotent power, which still has u and u' as fixed points. However, in an E-minimal algebra, every idempotent unary polynomial is either constant, or the identity map. Therefore p itself must be a permutation of $\mathbf{A}$.

We show that $k(\mathbf{x}, y)$ does not depend on $\mathbf{x}$. Suppose otherwise. Then there exist $\mathbf{b}$ and d such that $k(\mathbf{a}, d) \neq k(\mathbf{b}, d)$. But $k(\mathbf{a}, y)$ is a permutation, so we have $k(\mathbf{b}, d) = k(\mathbf{a}, d')$ for some $d' \in A$. Rectangularity implies that $k(\mathbf{b}, d) = k(\mathbf{a}, d)$, which is a contradiction.

Thus we have the identity $k(\mathbf{x}, y) = k(\mathbf{x}', y)$, and hence

$$m(\mathbf{x}, \mathbf{z}) = k(\mathbf{x}, m(\mathbf{y}, \mathbf{z})) = k(\mathbf{x}', m(\mathbf{y}, \mathbf{z})) = m(\mathbf{x}', \mathbf{z}) .$$

Thus $m(\mathbf{x}, \mathbf{z})$ does not depend on $\mathbf{x}$. This contradiction finishes the proof. □

One can avoid referring to the characterization of rectangular varieties, and can even localize the above theorem by an elementary, but longer argument. From Corollary 4.6 we get the following.

Corollary 4.11. *A strongly nilpotent E-minimal algebra generates a residually small variety if and only if it is essentially unary.*

5. A cardinality bound

This section is devoted to the proof that bounds the size of subdirectly irreducible algebras in weakly abelian E-minimal varieties.

Lemma 5.1. *Let X be a set, G an abelian group of exponent $m < \infty$, and $0 \neq a \in G$. Suppose that F is a set of functions mapping X to G such that for any two elements $c \neq d$ of X, the element a can be expressed in the form*

$$\pm(g_1(c) - g_1(d)) \pm (g_2(c) - g_2(d)) \pm \cdots \pm (g_n(c) - g_n(d)) ,$$

where each $g_j \in F$. Then $|X| \leq m^{|F|}$.

Proof. By Zorn's lemma, among the subgroups of G not containing the element a there is a maximal one, called H. We can replace G by G/H, and the element a by $a+H$, and the conditions will still hold. But the new G is now subdirectly irreducible. It is well-known that subdirectly irreducible abelian groups are cyclic or quasi-cyclic. As the exponent m is finite, we get that G is a cyclic group of exponent at most m. Hence $|G| \leq m$. The set of mappings in F clearly separates the points of X, that is, for every $c \neq d$ there exists an element $f \in F$ such that $f(c) \neq f(d)$. Therefore the mapping $c \mapsto (f(c) : f \in F)$ is injective from X to G^F, showing that $|X| \leq m^{|F|}$. □

Now we present our main lemma that bounds the size of subdirectly irreducibles. It shows clearly the advantage of E-minimal algebras over general nilpotent ones. The lemma works only if there is a "large" E-trace with a regular twin group containing a pair from the monolith. In the general case, there exist large E-traces with regular twin groups, but the monolith may restrict trivially to them (see [6] for details).

Lemma 5.2. *Let* $\mathbf{S}$ *be a finite weakly abelian subdirectly irreducible algebra, and* σ *a congruence of* $\mathbf{S}$ *such that* $\mathbf{S}/\sigma$ *is rectangular, and every term of* $\mathbf{S}/\sigma$ *depends on at most* r *variables.*

(1) *If* $\sigma = 0_{\mathbf{S}}$*, then* $|S| \le 2^M$*, where* $M = |\mathbf{F}_{\mathsf{V}(\mathbf{S})}(r+1)|$.
(2) *Suppose that* E *is an* E*-trace of* $\mathbf{S}$ *with respect to* σ *containing a nontrivial pair* (a,b) *in the monolith of* $\mathbf{S}$*, and that the group* $\mathbf{H} = \mathrm{Tw}(E, 1_{\mathbf{S}})$ *is transitive. Let* m *be the exponent of* $\mathbf{H}$*. Then*
$$|S| \le 2^M \cdot m^M \, .$$
(3) *If* $\mathbf{S}$ *is contained in a variety generated by an* n*-element algebra, then* $m \le n$.

Proof. Let $B = b/\sigma$. Consider all $r+1$-ary terms $t(x, \mathbf{y})$ such that the range of t on $\mathbf{S}$ intersects B. For every such term t_i fix a vector $\mathbf{u}^i$ such that $t_i(c, \mathbf{u}^i) \in B$ for a suitable $c \in S$. Here $i \in I$, where I is an at most M-element set. Define $f_i(x) = t_i(x, \mathbf{u}^i)$, and let F be the set of all functions f_i for $i \in I$. To every $c \in S$, assign the set
$$I(c) = \{ i \in I : t_i(c, \mathbf{u}^i) \in B \} \, .$$

Claim 5.3. *Suppose that* $(c,d) \in S^2$ *such that* $I(c) = I(d)$*, and* $t(x, \mathbf{y})$ *is any term such that* $t(c, \mathbf{v}) \in B$ *for some* $\mathbf{v}$*. Then there exists a vector* $\mathbf{u}$ *and an* $f \in F$ *such that* $f(x) = t(x, \mathbf{u})$ *holds for every* $x \in S$*, and all four vertices of the* $1_{\mathbf{A}}, 1_{\mathbf{A}}$*-matrix*
$$\begin{bmatrix} t(c,\mathbf{v}) & t(c,\mathbf{u}) \\ t(d,\mathbf{v}) & t(d,\mathbf{u}) \end{bmatrix}$$
are in B.

Proof. In the factor algebra $\mathbf{S}/\sigma$ the term $t(x, \mathbf{y})$ can depend on at most r variables in $\mathbf{y}$. Write $\mathbf{y} = (\mathbf{z}, \mathbf{z}')$ so that $\mathbf{z}$ is an r-tuple, and $t(x, \mathbf{y}) = t(x, \mathbf{z}, \mathbf{z}')$ does not depend on $\mathbf{z}'$ modulo σ. Using the same split we can write $\mathbf{v} = (\mathbf{w}, \mathbf{w}')$. Define $t'(x, \mathbf{z}) = t(x, \mathbf{z}, z_1, \dots, z_1)$ (where z_1 is the first component of $\mathbf{z}$). Then we have $t'(x, \mathbf{w}) \, \sigma \, t(x, \mathbf{v})$ for every x.

We assumed that $t(c, \mathbf{v}) \in B$. As B is a σ-class, this implies that $t'(c, \mathbf{w}) \in B$. Since t' is $r+1$-ary, there exists an $i \in I$ such that $t' = t_i$. Let $\mathbf{u} = (\mathbf{u}^i, u^i_1, \dots, u^i_1)$. Thus $f_i(x) = t_i(x, \mathbf{u}^i) = t'(x, \mathbf{u}^i) = t(x, \mathbf{u})$ for every x. From the definition of $\mathbf{u}^i$ we get that there is a $c' \in S$ such that $t_i(c', \mathbf{u}^i) \in B$. Thus we see that
$$t_i(c', \mathbf{u}^i) = t(c', \mathbf{u}) \, \sigma \, t(c, \mathbf{v}) \, ,$$

since both elements are in B. The rectangularity of $\mathbf{S}/\sigma$ gives

$$t(c,\mathbf{u})\ \sigma\ t(c,\mathbf{v})\,.$$

Hence $t_i(c,\mathbf{u}^i)=t(c,\mathbf{u})\in B$, and so $I(c)=I(d)$ implies that $t(d,\mathbf{u})=t_i(d,\mathbf{u}^i)\in B$. Thus

$$t(d,\mathbf{u})\ \sigma\ t(c,\mathbf{v})\,,$$

and by using the rectangularity of $\mathbf{S}/\sigma$ again we get that

$$t(d,\mathbf{u})\ \sigma\ t(d,\mathbf{v})\,.$$

This proves the claim. ∎

Consider any two elements $c\neq d$ such that $I(c)=I(d)$. As $\mathbf{S}$ is subdirectly irreducible, we have $(a,b)\in \mathrm{Cg}(c,d)$. Consider a Maltsev chain demonstrating this. The claim proves that the entire Maltsev chain is in B, since it starts from the element $b\in B$. In particular, we must have that $a\in B$. Thus, if $\sigma=0_{\mathbf{S}}$, then there is no such $c\neq d$, which means that the mapping $I(x)$, which assigns a subset of I to every element of S, is injective. Hence $|S|\le 2^{|I|}$, and (1) is proved.

To prove (2), let $E=e(A)\cap b/\sigma$ for an idempotent unary polynomial e. From the fact that $\mathbf{S}$ is weakly abelian we get, using Theorem 4.3, that E can be considered an abelian group with zero element b, such that the $1_{\mathbf{A}},1_{\mathbf{A}}$-matrices within E are exactly the matrices where the sum of the two diagonals is the same. In particular, if we prefix the elements of the $1_{\mathbf{A}},1_{\mathbf{A}}$-matrix in the claim by e we get

$$et(c,\mathbf{v})-et(d,\mathbf{v})=et(c,\mathbf{u})-et(d,\mathbf{u})=ef(c)-ef(d)\,.$$

Consider again the Maltsev-chain connecting a to b, which we have shown to be entirely in B. By applying e we may assume that this chain is entirely within E. If the elements of this chain are $a=a_0,a_1,\dots,a_n=b$, then as b is the zero element,

$$a=a-b=(a_0-a_1)+(a_1-a_2)+\cdots+(a_{n-1}-a_n)\,.$$

By the claim and the equation above, for each $1\le j\le n$ there is a $g_j\in F$ such that

$$a_{j-1}-a_j=\pm(eg_j(c)-eg_j(d))\,,$$

(the sign is $+$ or $-$ depending on whether (c,d) is mapped to (a_j,a_{j-1}) or (a_{j-1},a_j) via a unary polynomial). Hence,

$$a=\pm(eg_1(c)-eg_1(d))\pm(eg_2(c)-eg_2(d))\pm\cdots\pm(eg_n(c)-eg_n(d))\,.$$

To finish the proof, we use Corollary 5.1. Let $X\subseteq S$ be such that $I(c)$ is the same subset of I for every $c\in X$. Then

$$|X|\le m^{|I|}\le m^M\,.$$

This estimate holds for every class of the kernel of the mapping $I(x)$, which assigns a subset of I to every element of S. Hence,

$$|S|\le 2^{|I|}\cdot m^M\,,$$

proving (2).

Finally we prove (3). Suppose that $\mathbf{S} \in \mathsf{V}(\mathbf{A})$, where $\mathbf{A}$ is an n-element algebra. Let ℓ denote the least common multiple of the numbers $1, 2, \ldots, n$. Then for every unary function h on the set A we have that h^ℓ is idempotent, that is, $h^{2\ell} = h^\ell$. Now let $t(x, \mathbf{y})$ be any term in the language of $\mathbf{A}$, and write $t_{\mathbf{y}}(x) = t(x, \mathbf{y})$. Applying the previous observation to this unary function, where $\mathbf{y}$ is any vector in $\mathbf{A}$, we get that $t_{\mathbf{y}}^{2\ell}(x) = t_{\mathbf{y}}^{\ell}(x)$ is an identity of $\mathbf{A}$ (with variables x and $\mathbf{y}$). Hence this identity holds in $\mathbf{S}$, which clearly implies that the exponent m of $\mathbf{H}$ divides ℓ. We show that m is a prime power. Then $m \mid \ell$ implies that $m \leq n$, which is what we want to prove.

As E is an E-trace of $\mathbf{S}$ which contains a nontrivial pair from the monolith of $\mathbf{S}$ we see that $\mathbf{S}|_E$ is subdirectly irreducible. Apply (3) of Theorem 4.3 to E. We get that the module on E is also subdirectly irreducible. Its underlying group is finite, and is therefore the direct sum of its p-components for every prime divisor p of $|E|$. These are fully invariant subgroups, and therefore submodules. As this module is subdirectly irreducible, we get that the underlying group is a p-group for some prime p, and hence its exponent is a prime power indeed. □

Theorem 5.4. *Let* $\mathbf{A}$ *be a solvable* E*-minimal algebra of* n *elements. Then the following are equivalent.*

(1) $\mathsf{V}(\mathbf{A})$ *is residually small.*

(2) *Every subdirectly irreducible algebra in* $\mathsf{V}(\mathbf{A})$ *has at most* $(2n)^{n^{n^2}}$ *elements.*

(3) $\mathsf{V}(\mathbf{A})$ *is weakly abelian.*

Proof. We know that every solvable E-minimal algebra is barely nilpotent. Hence (1) $\Longrightarrow$(3) follows from Theorem 3.1. The implication (2) $\Longrightarrow$(1) is obvious. To prove (3) $\Longrightarrow$(2) let $\mathbf{S}$ be a finite subdirectly irreducible algebra in $\mathsf{V}(\mathbf{A})$. It is well-known and easy to see that all finite algebras in $\mathsf{V}(\mathbf{A})$ are E-minimal, and therefore so is $\mathbf{S}$. We shall set up the conditions of Lemma 5.2. Let σ be the partition given by the orbits of $\mathrm{Tw}(S, 1_{\mathbf{S}})$. By Corollary 4.9 we have that σ is a congruence, and $\mathbf{S}/\sigma$ is essentially unary, so we can take $r = 1$ in Lemma 5.2. Therefore $M \leq n^{n^2}$. If $\sigma = 0_{\mathbf{S}}$, then we are done. If $\sigma \neq 0_{\mathbf{S}}$, then it contains the monolith μ of $\mathbf{S}$. Take any nontrivial μ-block, and let E be the σ-class containing it. By the definition of σ, the set E is an orbit of $\mathbf{H} = \mathrm{Tw}(A, 1_{\mathbf{S}})$. Lemma 5.2 (3) shows that the exponent m of $\mathbf{H}$ is at most n. Thus we have proved that the bound in (2) works for every finite subdirectly irreducible. A well-known result of Quackenbush [10] shows that there are no infinite subdirectly irreducibles, and the proof is complete. □

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