Algebra Universalis, 32 (1994) 270–296
0002-5240/94/020270-27$01.50 + 0.20/0

Congruence semimodular varieties II: Regular varieties

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Dedicated to Bjarni Jónsson on the occasion of his 70th birthday

1. Introduction

In [8], P. Jones characterized the regular varieties of semigroups which are congruence semimodular and he partially solved the problem of characterizing the non-regular, congruence semimodular (CSM) varieties of semigroups. For regular varieties his characterization was an equational one, so necessarily a part of his argument was semigroup-theoretic. But some of his argument involved only congruence lattice manipulations and references to the two-element semilattice. It seemed plausible that one could give a universal algebraic characterization of all regular, CSM varieties.

Jones posed the problem of characterizing regular, CSM varieties at the International Conference on Universal Algebra and Lattice Theory held at Molokai, Hawaii in 1987. The authors were students attending that conference and became familiar with the problem. Agliano began a general investigation of CSM varieties under the supervision of his doctoral advisor, J. B. Nation, at the University of Hawaii. In the fall of 1988, Agliano filed his dissertation and Kearnes arrived at the University of Hawaii. We began discussing whether or not tame congruence theory could be applied to solve Jones' problem, at least for locally finite varieties or pseudo-varieties of finite algebras. This is the approach that had been suggested by Jones in 1987. We discovered a number of interesting facts about CSM varieties using tame congruence theory, but we did not solve Jones' problem at that time. (These "interesting facts" have since been collected in, *Congruence semimodular varieties I: locally finite varieties.*) After about two months our collaboration ended when Agliano returned to Italy.

For a while we felt compelled to publish some of our results on CSM varieties, but we were reluctant to do so without solving Jones' problem first. We renewed

Presented by G. McNulty.
Received September 26, 1990; accepted in final form December 2, 1991.

our collaboration in the fall of 1989 and finally solved Jones' problem. The solution is Theorem 3.3. About half of the argument was familiar to us from our 1988 discussions. In particular, we'd known for a year how to prove the crucial implication Theorem 3.3 $(1) \to (3)$. The turning point came when the reverse implication, Theorem 3.3 $(3) \to (1)$, was proved for finite algebras by using tame congruence theory. Then we made the exciting discovery that infinite algebras in regular, CSM varieties behave "as if they were finite." That is, we were able to extrapolate enough of the techniques of tame congruence theory to infinite algebras in regular, CSM varieties so that we could prove Theorem 3.3 $(3) \to (1)$ for any regular, CSM variety. The fact that this is possible affirms Jones' original insight that tame congruence theory could be used to solve the problem.

We feel that the solution to Jones' problem is the most important result in this paper. Its proof necessitates an examination of the one block property and urges further investigation of polynomially orderable varieties. We give these topics brief attention. Our conventions follow those of *Congruence semimodular varieties I: locally finite varieties*. Our reference for algebra is [10] and our reference for tame congruence theory (which we use very little of in this paper) is [6].

2. The one block property

We are interested in conditions strong enough to imply that an algebra, or every algebra in a variety, has a semimodular congruence lattice. We are interested mainly in algebras which are not congruence modular and our principal examples of these are semilattices and sets. In this section we isolate a certain congruence property common to semilattices and sets which is strong enough to force semimodular congruences.

DEFINITION 2.1. An algebra $\mathbf{A}$ is said to have the **one block property** (briefly OBP) if any atom $\theta \in \mathrm{Con}(\mathbf{A})$ has exactly one nontrivial congruence class. A class of similar algebras has the OBP if every member does.

THEOREM 2.2. *Lek $\mathcal{K}$ be a class of similar algebras closed under homomorphic images. Then $(1) \to (2) \to (3)$.*

(1) *$\mathcal{K}$ has the OBP.*

(2) *For any algebra $\mathbf{A} \in \mathcal{K}$, any atom $\alpha \in \mathrm{Con}(\mathbf{A})$ and any $\beta \in \mathrm{Con}(\mathbf{A})$, we have $\beta \circ \alpha \circ \beta = \alpha \vee \beta$.*

(3) *$\mathcal{K}$ is CSM.*

Proof. $(1) \to (2)$. Choose $\mathbf{A} \in \mathcal{K}$ and suppose that $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ and $\alpha \succ 0_A$. If $(a, b) \in \alpha \vee \beta$, then we can find a chain of elements $a = x_0, \ldots, x_n = b$ where

$(x_i, x_{i+1}) \in \alpha \cup \beta$ for each $i < n$. If this chain has minimal length, then the fact that α has only one nontrivial block implies that at most one nontrivial "α-link" is involved in this chain. From this it is clear that $\alpha \vee \beta = \beta \circ \alpha \circ \beta$.

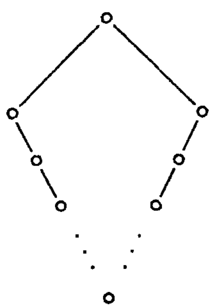
(2) $\to$ (3). Suppose that congruence semimodularity fails in $\mathscr{K}$. Then, there exists an algebra $\mathbf{A} \in \mathscr{K}$, an $\alpha \succ 0_A$, a $\beta \ngeq \alpha$ and a $\gamma \in \mathrm{Con}(\mathbf{A})$ with $\alpha \vee \beta > \gamma > \beta$. Pick $(a, b) \in \gamma - \beta$; we have $(a, b) \in \alpha \vee \beta = \beta \circ \alpha \circ \beta$ by hypothesis. Hence there are $u, v \in A$ with $a \,\beta\, u \,\alpha\, v \,\beta\, b$. This means that $u \,\beta\, a \,\gamma\, b \,\beta\, v$. But $\beta < \gamma$, so $(u, v) \in \alpha \wedge \gamma = 0_A$. Therefore $a \,\beta\, u \,0_A\, v \,\beta\, b$ and $(a, b) \in \beta$ which is a contradiction. □

We leave it to the reader to show that neither implication in Theorem 2.2 can be reversed.

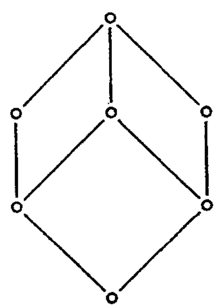
THEOREM 2.3. *Let $\mathscr{V}$ be a nontrivial variety which has the OBP. $\mathscr{V}$ is not congruence modular. In fact, for every nontrivial $\mathbf{A} \in \mathscr{V}$, either $\mathbf{K}$ or $\mathbf{D}_2$ occurs as a sublattice of* $\mathrm{Con}(\mathbf{B})$ *for some* $\mathbf{B} \le \mathbf{A}^2$.

Proof. Assume that $\mathbf{A}$ is a nontrivial member of $\mathscr{V}$, $\theta \in \mathrm{Con}(\mathbf{A})$ is compact and $\delta \in \mathrm{Con}(\mathbf{A})$ is a lower cover of θ. Let $\mathbf{B} \le \mathbf{A}^2$ be the subalgebra whose universe is θ. For $\alpha \in \mathrm{Con}(\mathbf{A})$ let α_0 denote the congruence on $\mathbf{B}$ consisting of all $((a, b), (c, d)) \in B^2$ such that $(a, c) \in \alpha$ and let α_1 denote the congruence consisting of all $((a, b), (c, d)) \in B^2$ such that $(b, d) \in \alpha$. Let $\bar{\alpha}$ denote $\alpha_0 \wedge \alpha_1$. From the definition of $\mathbf{B}$ we have $\theta_0 = \theta_1 = \bar{\theta}$ and $\delta_0, \delta_1 \prec \bar{\theta}$. Now, $\delta_0 \prec \bar{\theta}$, but $\bar{\delta} = \delta_0 \wedge \delta_1 \not\prec \bar{\theta} \wedge \delta_1 = \delta_1$, since $\delta_1/\bar{\delta} \in \mathrm{Con}(\mathbf{B}/\bar{\delta})$ has more than one nontrivial equivalence class. Hence, $\mathrm{Con}(\mathbf{B})$ is not even dually semimodular. This proves the first statement.

Since $\bar{\delta} \not\prec \delta_1$, we can find $\delta'_1 \in \mathrm{Con}(\mathbf{B})$ such that $\bar{\delta} < \delta'_1 < \delta_1$. Let $\delta'_0 = \{((a, b), (c, d)) \in B^2 \mid ((b, a), (d, c)) \in \delta'_1\}$. If $\delta'_0 \vee \delta'_1 < \bar{\theta}$, then $\delta'_0 \vee \delta'_1$, δ_0 and δ_1 generate a sublattice of $\mathrm{Con}(\mathbf{B})$ isomorphic to $\mathbf{D}_2$. Otherwise δ'_0, δ'_1 and δ_0 generate



K



$\mathbf{D}_2$

Figure 1

a pentagon in Con(**B**). By semimodularity, we cannot have $\bar{\delta} \prec \delta'_1$, so we can find δ''_1 such that $\bar{\delta} < \delta''_1 < \delta'_1$. Construct δ''_0 from δ''_1 in the same way that we constructed δ'_0 from δ'_1. Either $\delta''_0 \vee \delta''_1$, δ_0 and δ_1 generate a copy of $\mathbf{D}_2$ or else we can find a δ'''_1 and δ'''_0 with $\bar{\delta} < \delta'''_1 < \delta''_1 < \delta'_1$ and $\bar{\delta} < \delta'''_0 < \delta''_0 < \delta'_0$ as in our earlier argument. If this process fails to ever produce a copy of $\mathbf{D}_2$ as a sublattice of Con(**B**), then we end up constructing a copy of **K**. This proves the second statement. □

THEOREM 2.4. **A** *has the OBP if every finitely generated member of* $\mathbb{HS}(\mathbf{A})$ *has the OBP. In particular, a locally finite variety* $\mathcal{V}$ *has the OBP if and only if* $\mathcal{V}_{\text{fin}}$ *has the OBP.*

Proof. Assume that **A** fails to have the OBP. There is an atom $\alpha \in \text{Con}(\mathbf{A})$ such that $\alpha = \text{Cg}(a, b) = \text{Cg}(c, d) \neq \text{Cg}(a, c)$. Either $\text{Cg}(a, c) \cap \alpha = 0_A$ or $\text{Cg}(a, c) > \alpha$. We can find a finitely generated subalgebra $\mathbf{B} \le \mathbf{A}$ such that $(a, b) \in \text{Cg}^{\mathbf{B}}(c, d)$ and $(c, d) \in \text{Cg}^{\mathbf{B}}(a, b)$ and either $\text{Cg}^{\mathbf{B}}(a, c) \cap \text{Cg}^{\mathbf{B}}(a, b) = 0_B$ or $\text{Cg}^{\mathbf{B}}(a, c) > \text{Cg}^{\mathbf{B}}(a, b)$. Since $\text{Cg}^{\mathbf{B}}(a, b)$ is compact in Con(**B**) there is a $\beta \prec \text{Cg}^{\mathbf{B}}(a, b)$. Let $\mathbf{C} = \mathbf{B}/\beta$. Of course, $0_C \prec \text{Cg}^{\mathbf{C}}(a/\beta, b/\beta) = \text{Cg}^{\mathbf{C}}(c/\beta, d/\beta) \neq \text{Cg}^{\mathbf{C}}(a/\beta, c/\beta)$ so **C** fails to have the OBP. **C** is a finitely generated member of $\mathbb{HS}(\mathbf{A})$, so we're done. □

For each $n < \omega$, Example 7.1 of [4] describes a locally finite variety $\mathcal{V}$ which fails to have the OBP although $\mathbb{V}(\mathbf{F}_{\mathcal{V}}(n))$ does have the OBP.

THEOREM 2.5. $\mathcal{V}$ *has the OBP if and only if* $\mathbb{V}(\mathbf{AI}_N)$ *has the OBP for every* $\mathbf{A} \in \mathcal{V}$ *and every E-trace* $N \subseteq A$.

Proof. One direction of this theorem is trivial. A itself is an E-trace of **A** and $\mathbf{AI}_A$ is polynomially equivalent to **A**. Thus if $\mathbf{AI}_A$ has the OBP for every $\mathbf{A} \in \mathcal{V}$, then $\mathcal{V}$ has the OBP.

For the other direction assume that $\mathcal{V}$ has the OBP and that N is an E-trace of **A**. By Theorem 6.17 of [6], every $\mathbf{B} \in \mathbb{V}(\mathbf{AI}_N)$ is polynomially equivalent to $\mathbf{CI}_B$ for some E-trace $B \subseteq C$ where **C** is some member of $\mathbb{V}(\mathbf{A}) \subseteq \mathcal{V}$. Factoring by a congruence $\theta \in \text{Con}(\mathbf{C})$ maximal for $\theta|_B = 0_B$ and changing notation, we may assume that no nonzero congruence of **C** restricts trivially to B. The restriction map from Con(**C**) to Con($\mathbf{CI}_B$) is onto, so if $\beta \in \text{Con}(\mathbf{CI}_B)$ is an atom, then $\beta' = \text{Cg}^{\mathbf{C}}(\beta)$ is an atom of Con(**C**) which restricts to β. Since **C** has the OBP β' has only one block. This means that $\beta'|_B = \beta$ has only one block. Since β and **B** were arbitrary and $\mathbf{CI}_B$ and **B** are polynomially equivalent, $\mathbb{V}(\mathbf{AI}_N)$ has the OBP. □

COROLLARY 2.6. *If* $\mathcal{V}$ *is a locally finite variety with the OBP, then* $\text{typ}\{\mathcal{V}\} \subseteq \{\mathbf{0}, \mathbf{5}\}$.

Proof. Since $\mathscr{V}$ has the OBP, it is CSM. By Theorem 2.6 of [4], $\mathbf{1} \notin \operatorname{typ}\{\mathscr{V}\}$. We only need to show that **2**, **3** and **4** cannot occur in the type-set of a locally finite variety with the OBP. If some finite algebra $\mathbf{A}$ did have a minimal congruence β of type **2**, **3** or **4**, then for some $\langle 0_A, \beta\rangle$-trace N the algebra $\mathbf{AI}_N$ generates a nontrivial congruence modular variety which, by Theorem 2.5, has the OBP. This contradicts Theorem 2.3. □

3. Regular varieties and polynomially orderable algebras

The following definition is due to J. Plonka in [12].

DEFINITION 3.1. An equation $s(\bar{x}) \approx t(\bar{x})$ is **regular** if $s(\bar{x})$ and $t(\bar{x})$ have the same free variables. A variety $\mathscr{V}$ of algebras is **regular** if it can be axiomatized by regular equations.

We will say that a variety of algebras is **strongly irregular** if it satisfies an equation of the form $t(x, y) \approx x$, where y is a free variable of t.

For any similarity type τ we can define an algebra $\mathbf{S}_\tau$ in the following way. The universe of $\mathbf{S}_\tau$ is $\{0, 1\}$ and, for any n-ary operation symbol f, $f^{\mathbf{S}_\tau}$ is realized as

$$f^{\mathbf{S}_\tau}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i = 1 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

To make our discussion simpler, we will not deal with similarity types that have 0-ary fundamental operations. Now observe that if τ includes at least one operation symbol of arity ≥ 2, then $\mathbf{S}_\tau$ is term-equivalent to the two-element semilattice. Henceforth we will call this algebra the τ-**semilattice**. (If τ has only unary operations this name is a little misleading since in this case $\mathbf{S}_\tau$ is term-equivalent to the two-element set.)

It is not hard to prove that a variety $\mathscr{V}$ of type τ is regular if and only if $\mathbf{S}_\tau \in \mathscr{V}$. This is because $\mathbf{S}_\tau$ satisfies an equation ϵ in the language of $\mathscr{V}$ if and only if ϵ is regular. We define the **regularization** of $\mathscr{V}$ (**reg** $\mathscr{V}$) to be the variety generated by $\mathscr{V}$ and $\mathbf{S}_\tau$. Alternately, reg $\mathscr{V}$ is the variety axiomatized by the regular equations that hold in $\mathscr{V}$.

DEFINITION 3.2. If $\mathbf{A} \in \mathscr{V}$ and p is a unary polynomial of $\mathbf{A}$, then p is **permissible** if there is an $(n+1)$-ary term t which depends on all of its variables in $\mathscr{V}$ and an n-tuple $\bar{a} \in A^n$ such that $p(x) = t^{\mathbf{A}}(x, \bar{a})$. We define a quasi-order, $\ll_A$, on

A as follows: $\preccurlyeq_A$ is the transitive closure of

$$\{\langle p(a), a\rangle \mid a \in A \text{ and } p \text{ a permissible polynomial of } A\}.$$

Let $\approx_A$ be the equivalence relation on A defined by

$$a \approx_A b \leftrightarrow a \preccurlyeq_A b \quad \text{and} \quad b \preccurlyeq_A a.$$

$\mathbf{A}$ is called **polynomially orderable** if $\preccurlyeq_A$ is a partial ordering of A. $\mathscr{V}$ is called **polynomially orderable** if every member is. If $\alpha \in \mathrm{Con}(\mathbf{A})$ and $0 \in A$, then we will call 0 a **zero element for** α if the congruence class $0/\alpha$ is nontrivial and $(p(0), 0) \in \alpha$ implies $p(0) = 0$ for any permissible polynomial p. If $\alpha = 1_A$, we will call 0 a **zero element for A**.

Observe that every nonconstant unary polynomial is permissible. Further, in a regular variety the permissible polynomials are closed under composition, so $a \preccurlyeq_A b$ if and only if there is a permissible polynomial p such that $p(b) = a$. If $\mathscr{V}$ is regular, then p is a permissible polynomial of $\mathbf{A} \in \mathscr{V}$ if and only if $p(x) = t^{\mathbf{A}}(x, \bar{a})$ where $\bar{a} \in A^n$, t is an $(n+1)$-ary term of $\mathscr{V}$ which satisfies the bi-implication

$$t^{\mathbf{S}_\tau}(x_0, \ldots, x_n) = 1 \leftrightarrow x_i = 1 \qquad \text{for all } i.$$

The definition of "permissible" depends on $\mathscr{V}$ in that some terms may depend on more variables in an extension of $\mathscr{V}$ than they do in $\mathscr{V}$. However, among regular varieties this notion does not depend on $\mathscr{V}$; for in a regular variety every term t depends on every variable that occurs as a free variable in t. If there is a possibility of confusion we will specify the variety we are defining "permissible" relative to.

A zero element for $\mathbf{A}$ is just an element of $\mathbf{A}$ which is minimal under $\preccurlyeq_A$. If 0 is a zero element for $\mathbf{A}$, then the equivalence class $0/\approx_A$ is equal to $\{0\}$. If $\mathbf{A}$ belongs to a regular variety, a zero element for $\mathbf{A}$ is just an element of $\mathbf{A}$ which is an absorbing element for all the fundamental operations.

EXERCISE 1. Prove that if $\alpha \in \mathrm{Con}(\mathbf{A})$, then $\alpha \circ \preccurlyeq_A$ is a quasi-order on A. Show that $\preccurlyeq_{A/\alpha} = (\alpha \circ \preccurlyeq_A)/\alpha$.

EXERCISE 2. Prove that if $\mathbf{A}$ is locally finite and $\alpha \in \mathrm{Con}(\mathbf{A})$, then $\alpha \vee \approx_A = \alpha \circ \approx_A \circ \alpha$. Here the join is formed in $\mathrm{Eq}(\mathbf{A})$, the lattice of equivalence relations on A.

In these exercises the notion of permissible is defined relative to any variety containing $\mathbf{A}$. The results of these exercises are not used anywhere in this paper but they serve to show that the relations $\ll_A$ and $\sim_A$ are fairly well-behaved. For example, Exercise 3 implies that $\approx_A$ 4-permutes with every congruence on $\mathbf{A}$.

THEOREM 3.3. *If $\mathscr{V}$ is regular, the following are equivalent.*
(1) *$\mathscr{V}$ is congruence semimodular.*
(2) *$\mathscr{V}$ is congruence weakly semimodular.*
(3) *For all $\mathbf{A} \in \mathscr{V}$ and all nonzero $\alpha \in \mathrm{Con}(\mathbf{A})$ one has $\alpha \nsubseteq \approx_A$.*
(4) *For all $\mathbf{A} \in \mathscr{V}$ and all $a \neq b$ in A there is a unary polynomial p such that $p(a) \not\approx_A p(b)$.*
(5) *Every subdirectly irreducible algebra in $\mathscr{V}$ has the OBP and a zero element for the monolith.*

Proof. Actually we will show that conditions (3), (4) and (5) are equivalent for any $\mathscr{V}$ and that they imply (1) which obviously implies (2). We will use the regularity of $\mathscr{V}$ only to prove that (2) implies (3).

The equivalence of (3) and (4) is immediate since (4) says precisely that for all $a \neq b$ in A we have $\mathrm{Cg}^{\mathbf{A}}(a, b) \nsubseteq \approx_A$. Of course, this holds for all nonzero principal congruences if and only if it holds for all nonzero congruences.

To show that (3) is equivalent to (5) assume (3) and choose a subdirectly irreducible algebra $\mathbf{B} \in \mathscr{V}$ with monolith μ. Since $\mu \nsubseteq \approx_B$ we can pick $0, 1 \in B$ such that $\mu = \mathrm{Cg}(0, 1)$ where 1 is not $\ll_B 0$. If $b \in B$ is any element other than 0, then $(0, 1) \in \mathrm{Cg}(0, b)$. By Mal'cev's congruence generation theorem there is a unary polynomial p such that $p(0) \neq p(b)$ and either $p(0) = 1$ or $p(b) = 1$. Thus, $1 \ll_B 0$ or $1 \ll_B b$. The first case has been ruled out already, so $1 \ll_B b$. We cannot have $1 \ll_B b \ll_B 0$, so $b \ll_B 0$ is false for every b distinct from 0. This shows that 0 cannot be moved by any permissible polynomial; i.e., 0 is a zero element for $\mathbf{B}$. Since 0 belongs to a nontrivial μ-class it is a zero element for μ. Now μ is equal to the equivalence relation generated by $\{(p(0), p(1)) = (0, p(1)) \mid p \text{ permissible}\}$ and therefore the only nontrivial class of μ is the class containing 0. Hence (5) holds. Now assume that (5) holds and that (3) fails. First we will show that if $\mathscr{V}$ contains an algebra that fails condition (3), then we can find a subdirectly irreducible algebra in $\mathscr{V}$ which fails this condition. We will need the following result.

CLAIM. For any $\theta \in \mathrm{Con}(\mathbf{A})$ the equivalence relation $(\theta \vee \approx_A)/\theta$ is contained in $\approx_{A/\theta}$.

Proof of Claim. To see this, choose $x, y \in A$ such that $(x/\theta, y/\theta) \in (\theta \vee \approx_A)/\theta$. The pair (x, y) must lie in the transitive closure of $\theta \circ \approx_A$ which is contained in the transitive closure of $\theta \circ \ll_A$. Hence there is a chain of elements of A, $x = x_0, \ldots, x_n = y$ such that for every i either $(x_i, x_{i+1}) \in \theta$ or there is a permissible polynomial p_i such that $p_i(x_{i+1}) = x_i$. Factoring by θ this means that $x_0/\theta, \ldots, x_n/$

θ is a chain of elements where for each i either $x_i/\theta = x_{i+1}/\theta$ or there is a permissible polynomial $\bar{p}_i$ such that $\bar{p}_i(x_{i+1}/\theta) = x_i/\theta$. That is, $x/\theta = x_0/\theta \preccurlyeq_{A/\theta} x_n/\theta = y/\theta$. By symmetry, $y/\theta \preccurlyeq_{A/\theta} x/\theta$, so $x/\theta \approx_{A/\theta} y/\theta$.

Now assume that $\mathbf{A} \in \mathcal{V}$ has a nonzero congruence α for which $\alpha \subseteq \approx_A$. Since $\mathrm{Con}(\mathbf{A})$ is weakly atomic we can find δ and θ such that $0_A \leq \delta \prec \theta \leq \alpha \subseteq \approx_A$. In $\mathbf{A}/\delta$ we have $\theta/\delta \subseteq (\delta \vee \approx_A)/\delta$ and θ/δ is an atom in $\mathrm{Con}(\mathbf{A}/\delta)$. Replacing $\mathbf{A}$ by $\mathbf{A}/\delta$ and α by θ/δ and changing notation we may assume that α is an atom in $\mathrm{Con}(\mathbf{A})$. Now let β be a congruence on $\mathbf{A}$ which is maximal with respect to $\alpha \wedge \beta = 0_A$. Since α is an atom, β is completely meet-irreducible and has a unique upper cover β^*. In $\mathrm{Con}(\mathbf{B}/\beta)$ we have

$$\beta^*/\beta \subseteq (\beta \vee \alpha)/\beta \subseteq (\beta \vee \approx_A)/\beta \subseteq \approx_{A/\beta}.$$

Replacing $\mathbf{A}$ by $\mathbf{A}/\beta$ and changing notation one more time we may assume that $\mathbf{A}$ is subdirectly irreducible and that α is the monolith. Thus if (3) fails we can find a subdirectly irreducible algebra $\mathbf{A}$ with monolith α such that $\alpha \subseteq \approx_A$. But α has exactly one nontrivial block, by (5), and if 0 and 1 are elements such that $\alpha = \mathrm{Cg}(0, 1)$ and 0 is a zero element for α, then $0 \not\approx_A 1$. This contradicts our conclusion that $(0, 1) \in \alpha \subseteq \approx_A$. Hence (3) and (5) are equivalent.

In order to prove that (3) implies (1) we choose to first prove that (2) and (3) jointly imply (1). Then we will show that (3) implies (2). For the first step, assume (3) and deny (1) and we will argue that (2) must fail.

If (1) fails we can find an algebra $\mathbf{A} \in \mathcal{V}$ with $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ such that $\alpha \wedge \beta = 0_A \prec \alpha$ and $\beta < \gamma < \alpha \vee \beta$. Observe that, since $\mathrm{Con}(\mathbf{A})$ is upper continuous, the set

$$D = \{\delta \in [0_A, \beta] \mid \delta \prec \alpha \vee \delta\}$$

is closed under unions of chains. For suppose that $C \subseteq D$ is a chain and that $\bigvee_{\delta \in C} \delta = \theta \leq \beta$. Suppose also that $\theta \leq \psi < \alpha \vee \theta = \bigvee_{\delta \in C} \alpha \vee \delta$. Since $\alpha \nleq \psi$, for every $\delta \in C$ we must have $\delta \leq (\alpha \vee \delta) \wedge \psi < \alpha \vee \delta$, so $\delta = (\alpha \vee \delta) \wedge \psi$. Thus,

$$\theta = \bigvee_{\delta \in C} \delta = \bigvee_{\delta \in C} ((\alpha \vee \delta) \wedge \psi) = \left(\bigvee_{\delta \in C} (\alpha \vee \delta) \right) \wedge \psi = (\alpha \vee \theta) \wedge \psi = \psi,$$

so $\theta \prec \alpha \vee \theta$. Now, since D is closed under unions of chains we can use Zorn's Lemma to find an element $\nu \in [0_A, \beta]$ maximal for the property that $\nu \prec \alpha \vee \nu$. Replacing $\mathbf{A}$ by $\mathbf{A}/\nu$, α by $(\alpha \vee \nu)/\nu$ and β by β/ν and relabeling we may assume that for every nonzero $\delta \in [0_A, \beta]$ we have that $\alpha \wedge \delta = 0_A \prec \alpha$ and $\delta \not\prec \alpha \vee \delta$.

From here on our argument requires a good knowledge of the structure of a nontrivial α-class. In several places in the following arguments we will use the fact that if $(x, y) \in \mathrm{Cg}(w, z) - 0_A$, then $x \lessdot_A w$ or z and $y \lessdot_A w$ or z. The reason for this is that, by Mal'cev's congruence generation theorem, when $(x, y) \in \mathrm{Cg}(w, z) - 0_A$ there exists a unary polynomial p such that $p(w) \neq p(z)$ and $x = p(w)$ or $p(z)$. The polynomial p is permissible, so $x \lessdot_A z$ or w. Similarly, $y \lessdot_A z$ or w. Assume that $\alpha = \mathrm{Cg}(0, 1)$ where 1 is not $\lessdot_A 0$ (we can assume this since $\alpha \nsubseteq \approx_A$). Suppose that p is a unary polynomial such that $p(0) \neq p(1)$. Then $\alpha = \mathrm{Cg}(p(0), p(1))$ and therefore $(0, 1) \in \mathrm{Cg}(p(0), p(1))$. By the congruence generation theorem we can find a unary polynomial q such that $1 = qp(1) \neq qp(0)$. So far we have $\alpha = \mathrm{Cg}(0, 1) = \mathrm{Cg}(p(0), p(1)) = \mathrm{Cg}(qp(0), qp(1)) = \mathrm{Cg}(qp(0), 1)$. Hence, $0_A \leq \mathrm{Cg}(qp(0), 0) \leq \alpha$ and $0_a \prec \alpha$. If $0 \neq qp(0)$, then $(0, 1) \in \alpha = \mathrm{Cg}(0, qp(0))$. But this leads either to $1 \lessdot_A 0$ or $1 \lessdot_A qp(0) \lessdot_A 0$ which is false, so $qp(0) = 0$. This shows that whenever p is a unary polynomial such that $p(0) \neq p(1)$, then we can find a q such that $qp(0) = 0$ and $qp(1) = 1$. In particular, for any such p it must be that $0 \approx_A p(0)$ and $1 \approx_A p(1)$ since $0 = qp(0) \lessdot_A p(0) \lessdot_A 0$ and $1 = qp(1) \lessdot_A p(1) \lessdot_A 1$. Now suppose that $(x, y) \in \mathrm{Cg}(0, 1) - 0_A$. There is a unary polynomial p such that $p(0) \neq p(1)$ and $x = p(0)$ or $p(1)$. This implies that $x \approx_A 0$ or $x \approx_A 1$. Similarly, $y \approx_A 0$ or $y \approx_A 1$. Thus, any element of a nontrivial α-class is $\approx_A$-related to either 0 or 1. If x and y are distinct α-related elements it is impossible for both x and y to be $\approx_A$-related to 0 since $(0, 1) \in \alpha = \mathrm{Cg}(x, y)$ and we run into the contradiction that $1 \lessdot_A x \approx_A 0$ or $1 \lessdot_A y \approx_A 0$. In particular, the α-class containing 0 and 1 contains no element x distinct from 0 which satisfies $x \approx_A 0$. Thus, 0 is a zero element for α. Every element of $0/\alpha - \{0\}$ must be $\approx_A$-related to 1. If it turns out that 0 and 1 are incomparable under $\lessdot_A$, then by interchanging 0 and 1 and repeating the last few lines of the argument it follows that each nontrivial α-class contains exactly two elements; one $\approx_A$-related to 0 and one $\approx_A$-related to 1 and both elements are zero elements for α. In the case where 0 and 1 are $\lessdot_A$-comparable we must have $0 \lessdot_A 1$, since 1 is not $\lessdot_A 0$. This case must occur if any α-class contains more than two elements. Any other nontrivial α-class contains a pair $(0', 1')$ of the form $(p(0), p(1))$ where $0' \approx_A 0$ and $1' \approx_A 1$. To summarize, every nontrivial α-class is the union of two disjoint $(\alpha \cap \approx_A)$-classes and at most one of the $(\alpha \cap \approx_A)$-classes is nontrivial. We will call $\{x, y\}$ a $\langle 0_A, \alpha\rangle$-**pseudotrace** if $(x, y) \in \alpha - \approx_A$. (We leave it to the interested reader to show that if **A** is finite, then a $\langle 0_A, \alpha\rangle$-pseudotrace is actually a $\langle 0_A, \alpha\rangle$-trace in the sense of tame congruence theory. This fact is not important to us, but it explains our terminology.) From what we have said, two α-related elements constitute a $\langle 0_A, \alpha\rangle$-pseudotrace if and only if one of them is a zero element for α. We have also shown that if $\{x, y\}$ is a $\langle 0_A, \alpha\rangle$-pseudotrace and the $(\alpha \cap \approx_A)$-class containing y is nontrivial, then $x \lessdot_A y$ and x is a zero element for α. Any two $\langle 0_A, \alpha\rangle$-pseudotraces are polynomially isomorphic; therefore, if

$\{x, y\}$ and $\{x', y'\}$ are $\langle 0_A, \alpha\rangle$-pseudotraces, then either $x \approx_A x'$ and $y \approx_A y'$ or else $x \approx_A y'$ and $y \approx_A x'$. Each α-block of more than two elements contains a unique zero element for α so any two α-related elements can be connected by a chain of at most two $\langle 0_A, \alpha\rangle$-pseudotraces. Any element common to two distinct $\langle 0_A, \alpha\rangle$-pseudotraces must be a zero element for α and also must be $\leqslant_A$ any α-related element.

Figure 2 illustrates what we have discovered about the α-classes and the $\approx_A$-classes of **A**. Elements of **A** are denoted by small circles in this figure. Line segments connecting elements indicate which two-element subsets are $\langle 0_A, \alpha\rangle$-pseudotraces. The solid boxes indicate the partition of A induced by α and the dashed ovals indicate the partition of A induced by $\approx_A$. The figure on the left illustrates the case where the two elements of a $\langle 0_A, \alpha\rangle$-pseudotrace are comparable while the figure on the right illustrates the case where they are not.

Now we return to the proof that if (3) holds and Con(**A**) is not semimodular, then Con(**A**) is not even weakly semimodular. Recall that $\beta < \gamma < \alpha \vee \beta$. If $\gamma/\beta \subseteq (\beta \vee \approx_A)/\beta \subseteq \approx_{A/\beta}$, then the algebra $\mathbf{A}/\beta$ fails condition (3). Hence there is a pair of elements $(u, v) \in \gamma - (\beta \vee \approx_A)$ and a chain of elements $u = x_0, x_1, \ldots, x_n = v$ with the property that, for each $i < n$, $\{x_i, x_{i+1}\}$ is a $\langle 0_A, \alpha\rangle$-pseudotrace or $(x_i, x_{i+1}) \in \beta$. We may assume that (u, v) and the chain connecting them have been chosen so that no other pair of elements in $\gamma - (\beta \vee \approx_A)$ can be connected by a shorter chain of the same type. If $(x_0, x_1) \in \beta$, then $(x_1, x_n) \in \gamma - (\beta \vee \approx_A)$ and x_1 and x_n are connected by a chain shorter than the one connecting x_0 and x_n; this is impossible by our assumption. Therefore, $\{x_0, x_1\}$ is a $\langle 0_A, \alpha\rangle$-pseudotrace. Similarly $\{x_{n-1}, x_n\}$, is a $\langle 0_A, \alpha\rangle$-pseudotrace. We have $x_0 = u \not\approx_A v = x_n$, so $\{x_0, x_1\}$ is a $\langle 0_A, \alpha\rangle$-pseudotrace which has an element $\approx_A$-related to u and $\{x_{n-1}, x_n\}$ is a $\langle 0_A, \alpha\rangle$-pseudotrace which has an element $\approx_A$-related to v. It follows that every $\langle 0_A, \alpha\rangle$-pseudotrace contains exactly one element $\approx_A$-related to u and exactly one element $\approx_A$-related to v. Since (x_0, x_1), $(x_{n-1}, x_n) \in \alpha$ and $(u, v) \notin \alpha$ there is a value of i, $1 < i < n-2$, such that $(x_i, x_{i+1}) \in \beta$. Both x_i and x_{i+1} belong to $\langle 0_A, \alpha\rangle$-pseudotraces, so $x_i \approx_A u$ or v and $x_{i+1} \approx_A u$ or v. We cannot have $u \approx_A x_i \beta x_{i+1} \approx_A v$ or $u \approx_A x_{i+1} \beta x_i \approx_A v$ for these contradict the fact that $(u, v) \notin \beta \vee \approx_A$. Thus, $x_i \approx_A u \approx_A x_{i+1}$ or

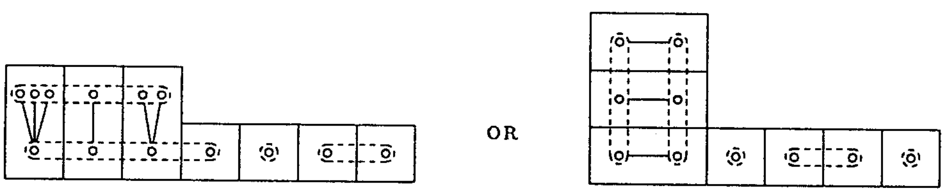


Figure 2

$x_i \approx_A v \approx_A x_{i+1}$. Our argument has been symmetric in u and v up to this point, so we may assume that v is not $\ll_A u$. We have shown that there exists a pair $(x_i, x_{i+1}) \in \beta - 0_A$ where $x_i \approx_A x_{i+1} \approx_A u$ or v. If there exist a pair $(x, y) \in \beta - 0_A$ with $x \approx_A u \approx_A y$, then let $(a, b) = (x, y)$. If there is no such pair, then we must have $x_i \approx_A x_{i+1} \approx_A v$ and in this case we let $(a, b) = (x_i, x_{i+1})$. Choose $(c, d) \in \mathrm{Cg}(a, b) - \approx_A$ and let $\beta' = \mathrm{Cg}(c, d)$.

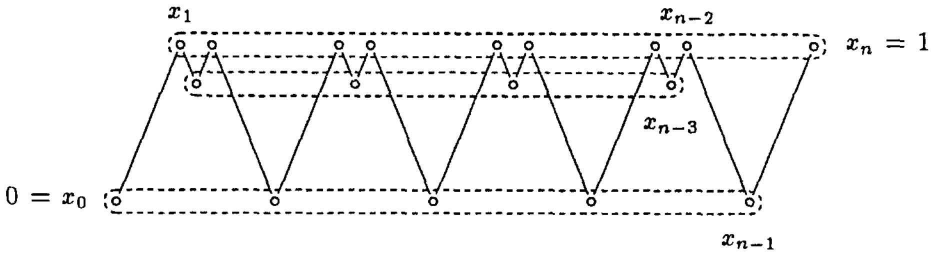
CLAIM. $0_A \prec \beta' \le \beta$.

Proof of Claim. Pick $(e, f) \in \beta' - 0_A$ and let $\beta'' = \mathrm{Cg}(e, f) \le \beta'$. We can establish the claim by showing that $\beta'' = \beta'$. Since every nonzero congruence contains a pair of $\approx_A$-inequivalent elements we may assume that $e \not\approx_A f$. Now, there is a $\gamma'' \in \mathrm{Con}(\mathbf{A})$ such that $\beta'' < \gamma'' < \alpha \vee \beta''$. Choose $(u'', v'') \in \gamma'' - (\beta'' \vee \approx_A)$ and a chain $u'' = y_0, \ldots, y_m = v''$ in which each link is either a $\langle 0_A, \alpha\rangle$-pseudotrace or a pair of β''-related elements. We may assume that u'', v'' and the chain are chosen so that the chain is of minimal length for all such chains connecting pairs in $\gamma'' - (\beta'' \vee \approx_A)$. As in our earlier argument, u'' and v'' represent the two distinct $\approx_A$-classes found in any $\langle 0_A, \alpha\rangle$-pseudotrace. We can assume that $u \approx_A u''$ and $v \approx_A v''$. Also, as the earlier argument showed, we can find a j such that $(y_j, y_{j+1}) \in (\beta'' \cap \approx_A)$ and $y_j \approx_A u \approx_A y_{j+1}$ or $y_j \approx_A v \approx_A y_{j+1}$. Now $(y_j, y_{j+1}) \in \mathrm{Cg}(a, b)$, so $y_j \approx_A y_{j+1} \ll_A a \approx_A b$. It is impossible for $v \approx_A y_j \ll_A a \approx_A u$, since v is not $\ll_A u$. Hence if $a \approx_A u$, then $y_j \approx_A u$. On the other hand, if $a \approx_A v \approx_A b$, then there is no pair $(x, y) \in \beta - 0_A$ for which $x \approx_A u \approx_A y$, so in this case $y_j \approx_A v \approx_A y_{j+1}$. This shows that $y_j \approx_A a \approx_A b \approx_A y_{j+1}$. Since $\mathrm{Cg}(y_j, y_{j+1}) \le \mathrm{Cg}(e, f) \le \mathrm{Cg}(c, d) \le \mathrm{Cg}(a, b)$, we get that $y_j \approx_A y_{y+1} \ll_A c$ or d, $y_j \approx_A y_{j+1} \ll_A e$ or f and $c, d, e, f \ll_A a \approx_A b$. At least one of c or d and one of e or f must be $\approx_A$-related to $y_j \approx_A a$; say $c \approx_A e \approx_A a \approx_A y_j$. Then, since $c \not\approx_A d$ and $d \ll_A a \approx_A c$, we get that d is strictly $\ll_A c$. Similarly, f is strictly $\ll_A e$. (Since the pair (e, f) was an arbitrary member of $\beta' - \approx_A$ we have shown that one of the coordinates of any $(r, s) \in \beta' - \approx_A$-related to $a \approx_A y_j$. This fact will be used shortly.) We can find a unary polynomial p that is a composition of permissible polynomials such that $p(e) = c$ since $c \approx_A e$. If $p(f) = d$, then $\beta' = \mathrm{Cg}(c, d) = \mathrm{Cg}(p(e), p(f)) \le \mathrm{Cg}(e, f) = \beta''$ and we are done. In the alternate case we have that (c, d), $(e, f) \in \beta' - 0_A$, $p(e) = c$ and $p(f) \ne d$, so $(p(f), d) \in \beta' - 0_A$ and both d and $p(f)$ $(\ll_A f)$ are strictly $\ll_A p(e) = c \approx_A a \approx_A y_j$. But now if $(g, h) \in \mathrm{Cg}(p(f), d) - \approx_A$, then each of g and h is $\ll_A d$ or $p(f)$. This shows that both g and h are both strictly $\ll_A a \approx_A y_j$. This contradicts our last parenthetical remark; one of the coordinates must be $\approx_A$-related to a and to y_j. This contradiction proves that $p(f) = d$ and hence that $\beta' = \beta''$.

Since $\beta' \in [0_A, \beta]$ we must have $\alpha \wedge \beta' = 0_A \prec \alpha$ and $\beta' \not\prec \alpha \vee \beta'$. Replace β by β' and change notation so that now $0_A \prec \alpha, \beta$ and $\beta < \gamma < \alpha \vee \beta$. This is a failure of weak semimodularity in Con(**A**). Thus we have proved that (2) and (3) jointly imply (1) as we promised. Now we show that (3) implies (2) which simultaneously establishes that (3) implies (1). Recall that our deductions about the structure of a nontrivial α-class were only based on the facts that $\alpha \succ 0_A$ and $\alpha \nsubseteq \approx_A$. Since we are now assuming that $\beta \succ 0_A$ and $\beta \nsubseteq \approx_A$ the same conclusions are true for the structure of a nontrivial β-class. For the rest of the proof we will use these facts and also the term $\langle 0_A, \beta \rangle$-**pseudotrace** without comment.

Choose $(0, 1) \in \gamma - (\beta \vee \approx_A)$ and a chain of elements $0 = x_0, x_1, \ldots, x_n = 1$ with the property that, for each $i < n$, $\{x_i, x_{i+1}\}$ is a $\langle 0_A, \alpha \rangle$-pseudotrace or a $\langle 0_A, \beta \rangle$-pseudotrace. With a now-familiar argument, we may assume that (0, 1) and the chain connecting them have been chosen so that no other pair of elements in $\gamma - (\beta \vee \approx_A)$ can be connected by a shorter chain of the same type. As before, $\{x_0, x_1\}$ and $\{x_{n-1}, x_n\}$ must be $\langle 0_A, \alpha)$-pseudotraces. $x_0 = 0 \not\approx_A 1 = x_n$ so each $\langle 0_A, \alpha \rangle$-pseudotrace contains one element $\approx_A$-related to 0 and one element $\approx_A$-related to 1. In particular, since $x_0 = 0$ and $\{x_0, x_1\}$ is a $\langle 0_A, \alpha \rangle$-pseudotrace we must have $x_1 \approx_A 1$. Similarly, $x_{n-1} \approx_A 0$. Now, if $\{x, y\}$ is a $\langle 0_A, \beta \rangle$-pseudotrace and $0 \approx_A x$ and $1 \approx_A y$, then $0 \approx_A x \,\beta\, y \approx_A 1$, a contradiction. Thus we can assume that, say, 0 is not $\approx_A$-related to any element in any $\langle 0_A, \beta \rangle$-pseudotrace. It follows that x_{n-1} lies in no $\langle 0_A, \beta \rangle$-pseudotrace, so the second to last link in the chain must also be a $\langle 0_A, \alpha \rangle$-pseudotrace. But now $\{x_{n-2}, x_{n-1}\}$ and $\{x_{n-1}, x_n\}$ are overlapping $\langle 0_A, \alpha \rangle$-pseudotraces so, recalling the structure of a nontrivial α-class, $x_{n-1} \leqslant_A x_{n-2} \approx_A x_n = 1$. The third to last link cannot also be a $\langle 0_A, \alpha \rangle$-pseudotrace, for then our chain would end with three consecutive $\langle 0_A, \alpha \rangle$-pseudotraces. The minimality of our chain forbids three consecutive $\langle 0_A, \alpha \rangle$-pseudotraces because every pair of α-related elements can be connected by a chain of most two $\langle 0_A, \alpha \rangle$-pseudotraces. Thus $\{x_{n-3}, x_{n-2}\}$ is a $\langle 0_A, \beta \rangle$-pseudotrace with $x_{n-2} \approx_A 1$. This shows that $x_{n-3} \not\approx_A 1$ and of course $x_{n-3} \not\approx_A 0$ since no member of a $\langle 0_A, \beta \rangle$-pseudotrace is $\approx_A$-related to 0. From this we get that x_{n-3} is not $\approx_A$-related to any member of any $\langle 0_A, \alpha \rangle$-pseudotrace. The fourth to last link in our chain must be a $\langle 0_A, \beta \rangle$-pseudotrace. Now x_{n-3} belongs to overlapping $\langle 0_A, \beta \rangle$-pseudotraces, so $x_{n-3} \leqslant_A x_{n-4} \approx_A x_{n-2}$. Combining this with our earlier deduction yields $x_{n-3} \beta x_{n-2} \,\alpha\, x_{n-1}$ and $x_{n-3}, x_{n-1} \leqslant_A x_{n-2}$. See Figure 3.

We can find a polynomial p which is a composition of permissible polynomials such that $p(x_{n-2}) = x_{n-1}$. If $p(x_{n-2}) \neq p(x_{n-3})$, then $x_{n-2} \approx_A p(x_{n-2}) = x_{n-1}$ which we know to be false ($\{x_{n-2}, x_{n-1}\}$ is a pseudotrace). Thus, $p(x_{n-2}) = p(x_{n-3}) = x_{n-1}$ and $x_{n-1} \leqslant_A x_{n-3}$. By a symmetric argument we can find a composition of permissible polynomials, q, such that $q(x_{n-2}) = x_{n-3}$ and get $q(x_{n-1}) = x_{n-3}$ implying that $x_{n-3} \leqslant_A x_{n-1}$. Thus, $x_{n-3} \approx_A x_{n-1} \approx_A 0$. This is a contradiction to our earlier conclusion that x_{n-3} is not $\approx_A$-related to any member of a $\langle 0_A, \alpha \rangle$-pseudotrace. This contradiction finishes the proof that (3) implies (2) and also the proof that (3) implies (1).



ELEMENTS OF THE SAME HEIGHT ARE $\approx_A$-RELATED

Figure 3

To prove that (2) implies (3) for regular varieties let's assume that (3) fails and prove that (2) must also fail. As we have already seen, since $\mathscr{V}$ fails (3) we can find a subdirectly irreducible algebra $\mathbf{A} \in \mathscr{V}$ with monolith α such that $\alpha \subseteq \approx_A$. Let τ denote the type of $\mathscr{V}$ and let $\mathbf{S} = \mathbf{S}_\tau$ be the τ-semilattice. We will prove that the congruence lattice of $\mathbf{A} \times \mathbf{S}$ is not weakly semimodular. For this argument we will write *as* instead of (a, s) to denote a member of $\mathbf{A} \times \mathbf{S}$. Choose $(a, b) \in \alpha - 0_A$ and let $\xi = \mathrm{Cg}^{\mathbf{A} \times \mathbf{S}}(a1, a0)$, $\lambda = \mathrm{Cg}^{\mathbf{A} \times \mathbf{S}}(a0, b0)$ and $\psi = \mathrm{Cg}^{\mathbf{A} \times \mathbf{S}}((a0, b0), (a1, b1))$. We will show that the sublattice of $\mathrm{Con}(\mathbf{A} \times \mathbf{S})$ generated by ξ, λ and ψ is the lattice in Figure 4.

Notice that if $x \ll_A y$ then, since there is a term t depending on all of its variables such that $x = t^{\mathbf{A}}(y, \bar{u})$, we get that $(x1, x0) \in \mathrm{Cg}^{\mathbf{A} \times \mathbf{S}}(y1, y0)$ because for $s = 0$ or 1 we have $xs = t^{\mathbf{A}}(ys, u_0 1, u_1 1, \ldots, u_n 1)$. In particular, since $(a, b) \in \alpha \subseteq \approx_A$ we get $b \ll_A a$ and so $(b1, b0) \in \mathrm{Cg}(a1, a0) = \xi$. If π_A and π_S denote the kernels of the coordinate projections, then $\xi \le \pi_A$ and $\lambda < \psi \le \pi_S$, so $0_{A \times S} = \xi \wedge \lambda = \xi \wedge \psi$. Also, $(a1, b1) \in \xi \circ \lambda \circ \xi$, so $\psi \le \xi \vee \lambda$. The congruence λ equals $\{(x0, y0) \mid (x, y) \in \alpha\} \cup 0_{A \times S}$ which covers $0_{A \times S}$. If $0_{A \times S} = \xi \wedge \lambda \prec \xi$, then since $\lambda < \psi < \lambda \vee \xi$ we have a failure of weak semimodularity in $\mathrm{Con}(\mathbf{A} \times \mathbf{S})$. In

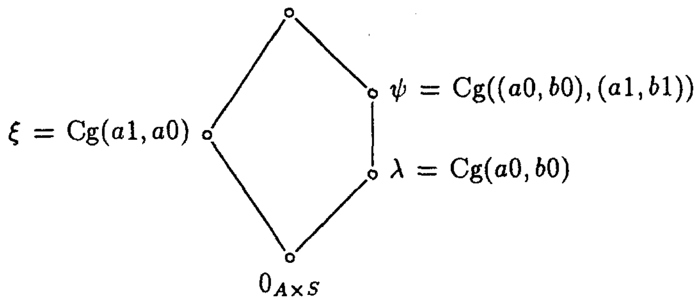


Figure 4

this case we would be finished. Thus, assume for now that $0_{A \times S} \not\prec \xi$ and choose a (principal) congruence θ such that $0_{A \times S} < \theta < \xi$. Necessarily θ is of the form $\mathrm{Cg}(c1, c0)$. Since $(c1, c0) \in \xi$ we can use Mal'cev's congruence generation theorem to find that there is a nonconstant unary polynomial p of $\mathbf{A} \times \mathbf{S}$ such that $p(a1) = c1$ or $p(a0) = c1$. Examining first coordinates shows that there is a permissible polynomial $\bar{p}$ of $\mathbf{A}$ such that $c = \bar{p}(a)$. Hence, $c \ll_A a$. It is not true that $a \approx_A c$ or else we get that $a \ll_A c$ and $(a1, a0) \in \mathrm{Cg}(c1, c0) < \mathrm{Cg}(a1, a0)$. If d is any other element for which $d \ll_A a$ but $d \not\approx_A a$, then $a/\mathrm{Cg}^{\mathbf{A}}(c, d) = \{a\}$, since no nonconstant (or even permissible) polynomial can map either c or d to a. But the subdirect irreducibility of $\mathbf{A}$ implies that $c = d$, since $\{a, b\} \subseteq a/\gamma$ whenever $\gamma \in \mathrm{Con}(\mathbf{A})$ is nonzero. Thus, c is the unique element of A which is strictly $\ll_A a$. It is therefore impossible to move c with a permissible polynomial, for $p(c) \ll_A c$ when p is permissible; i.e. c is a zero element for $\mathbf{A}$. From the uniqueness of c it follows that $\theta = \mathrm{Cg}(c1, c0)$ is the unique nonzero congruence below ξ, so $0_{A \times S} \prec \theta \prec \xi$, and that θ collapses only two distinct elements; θ is equal to the equivalence relation generated by $(c1, c0)$. Let $\lambda' = \lambda \vee \theta$ and $\psi' = \psi \vee \theta$. Since $0_{A \times S} \prec \theta, \lambda$ we must have $\theta \prec \lambda \vee \theta = \lambda'$ or else we have found a failure of weak semimodularity and we are done. Thus, assume that $\theta \prec \lambda'$. Notice that $(a1, b1) \notin \lambda' = \mathrm{Cg}((c1, c0), (a0, b0))$ since no nonconstant unary polynomial can map any element with first coordinate c to any element with first coordinate a nor any element with second coordinate 0 to any element with second coordinate 1. We have $\lambda' < \lambda' \vee \mathrm{Cg}(a1, b1) = \psi'$ and $\xi \nleq \lambda'$. This gives us that $\theta = \xi \wedge \lambda' \prec \xi$ and $\lambda' < \psi' \leq \xi \vee \lambda'$. See Figure 5.

If we can show that $\psi' \neq \xi \vee \lambda'$, then we will have $\theta \prec \xi, \lambda'$ but $\lambda' \not\prec \xi \vee \lambda'$ and we will have found a failure of weak semimodularity in $\mathrm{Con}(\mathbf{A} \times \mathbf{S})$.

We need to show that

$$\psi' = \mathrm{Cg}((a0, b0), (a1, b1), (c1, c0)) \neq \mathrm{Cg}((a1, a0), (a0, b0), (c1, c0)) = \xi \vee \lambda'.$$

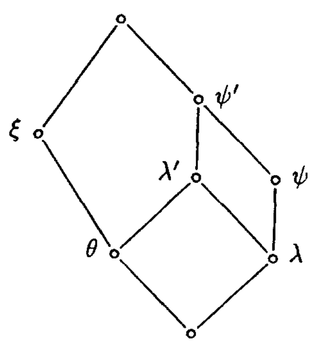


Figure 5

This is equivalent to showing that $(a1, a0) \notin \psi'$. Since $\alpha \subseteq \approx_A$ and c is a zero element, $c/\alpha \subseteq c/\approx_A = \{c\}$. Now, for any polynomial p we have $(p(a), p(b)) \in \alpha$ since $(a, b) \in \alpha$, so if $p(a) = c$, then $p(b) \in c/\alpha = \{c\}$. Similarly, if $p(b) = c$, then $p(a) = c$. It follows that for any polynomial of $\mathbf{A} \times \mathbf{S}$ we have $p(as) = ct$ if and only if $p(bs) = ct$ when $s, t \in S$. By Mal'cev's congruence generation theorem we find that $\psi = \mathrm{Cg}((a0, b0), (a1, b1), (c1, c0)) = \mathrm{Cg}((a0, b0), (a1, b1)) \cup \mathrm{Cg}((c1, c0)) = \psi \cup \theta$. Since $(a1, a0) \notin \psi$ and $(a1, a0) \notin \theta$ we obtain the desired conclusion that $(a1, a0) \notin \psi'$. This shows that if $\alpha \subseteq \approx_A$, then $\mathbf{A} \times \mathbf{S}$ is not CWSM and finishes the proof of the theorem. □

Before leaving this proof, let us make a few remarks about the concept of a permissible polynomial. Early in our investigations we thought it natural to deal only with nonconstant unary polynomials in our arguments, but we ran into problems. Compositions of nonconstant polynomials may be constant; unary polynomials may be nonconstant on $\mathbf{A}$, but constant on a subalgebra or homomorphic image of $\mathbf{A}$. It was clear to us that we needed to permit some constant polynomials in our definitions of $\preccurlyeq_A$ and $\approx_A$. This led us to the definition of permissible polynomials. With hindsight, we now see that when working with regular varieties this definition is not necessary. The polynomial $p(x) = t^{\mathbf{A}}(x, a_1, \ldots, a_n)$ is permissible for $\mathbf{A}$ if and only if the polynomial $p'(x) = t^{\mathbf{A} \times \mathbf{S}}(xy, a_1 1, \ldots, a_n 1)$ is nonconstant on $\mathbf{A} \times \mathbf{S}_\tau$. We could replace all occurrences of $\mathbf{A}$ in our argument with $\mathbf{A} \times \mathbf{S}_\tau$ and argue with nonconstant unary polynomials in place of permissible polynomials. To do this, however, would be to complicate a proof that is complicated enough. Therefore, we decided to retain the notion of permissible polynomial.

Several interesting points about regular, CSM varieties follow from Theorem 3.3. We separate these points out in the following corollaries.

COROLLARY 3.4. *If $\mathcal{V}$ is a regular, CSM variety, then* $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{0}, \mathbf{5}\}$.

Proof. If $\mathbf{A} \in \mathcal{V}$ is finite, $\alpha \in \mathrm{Con}(\mathbf{A})$ is an atom and N is any $\langle 0_A, \alpha \rangle$-trace of type different than $\mathbf{0}$ or $\mathbf{5}$, then from the nonconstant unary and binary polynomials of $\mathbf{A}|_N$ one can derive permissible polynomials which act transitively on N. Hence $N^2 \subseteq \approx_A$. Each nontrivial α-class is connected by $\langle 0_A, \alpha \rangle$-traces, so $\alpha \subseteq \approx_A$, which contradicts Theorem 3.3. □

Since $\mathrm{typ}\{\mathcal{V}\} \subseteq \mathrm{typ}\{\mathrm{reg}\,\mathcal{V}\}$, it follows that if $\mathrm{typ}\{\mathcal{V}\} \nsubseteq \{\mathbf{0}, \mathbf{5}\}$, then $\mathcal{V}$ is contained in no regular, CSM variety.

COROLLARY 3.5. *If $\mathcal{V}$ is a regular, CSM variety, then $\mathcal{V}$ contains no nontrivial strongly irregular subvarieties.*

Proof. Suppose that $\mathbf{A} \in \mathscr{W} \subseteq \mathscr{V}$ where $\mathscr{W}$ is strongly irregular and $|A| > 1$. Let $t(x, y)$ be the binary term witnessing strong irregularity for $\mathscr{W}$. Then, for any $a, b \in A$, set $p_a(x) = t^{\mathbf{A}}(a, x)$. Then we have $p_a(b) = t^{\mathbf{A}}(a, b) = a$ and $p_a(x)$ is permissible relative to $\mathscr{V}$, so $a \ll_A b$. Since a and b were arbitrary, $\approx_A = 1_A$. This contradicts Theorem 3.3. □

DEFINITION 3.6. A **dimension function** for a complete lattice $\mathbf{L}$ is a function $\delta: L \to \omega + 1$ defined as follows: for $x \in L$, $\delta(x) = n$ if every maximal chain in $I = [0_L, x]$ has length n. We define $\delta(x) = \omega$ if I has no finite maximal chain.

If $\mathbf{L}$ has an element x such that $[0_L, x]$ has finite maximal chains, but not all maximal chains have the same length, then $\mathbf{L}$ does not have a dimension function.

COROLLARY 3.7. *If $\mathscr{V}$ is a regular variety, then $\mathscr{V}$ is CSM if and only if for all $\mathbf{A} \in \mathscr{V}$* Con(**A**) *has a dimension function.*

Proof. Every complete, semimodular lattice has a dimension function satisfying very special properties (see Theorem 3.10 of [5], for example). Therefore, the forward implication of the corollary is true. The argument for the reverse implication can be copied from our proof that (2) implies (3) in Theorem 3.3. We leave this chore to the reader. The idea is to assume that (3) fails and then argue just as we did in Theorem 3.3 except where we used weak semimodularity show that the existence of a dimension function suffices. □

EXAMPLE 3.8. We will show that regular, CSM varieties are not characterized by their type-set, thus proving that the converse of Corollary 3.4 fails. First, we give an example of a regular, unary variety with type-set $\{\mathbf{0}\}$. Let $\mathscr{V}$ be the unary variety with two fundamental operations f and g satisfying the following identities:

$$f^2(x) \approx f(x) \qquad g^2(x) \approx g(x) \qquad fgfg(x) \approx fg(x) \qquad gfgf(x) \approx gf(x).$$

$\mathscr{V}$ is regular and the reader can easily check that every unary polynomial on $\mathbf{A} \in \mathscr{V}$ is idempotent. This implies that $\text{typ}\{\mathscr{V}\} = \{\mathbf{0}\}$. On the other hand, if $\mathbf{B} = \langle \{0, 1\}; f, g \rangle$ where $f^{\mathbf{B}}(x) = 0$ and $g^{\mathbf{B}}(x) = 1$, then $\mathbf{B}$ is a simple member of $\mathscr{V}$ for which $\approx_B = 1_B$ (we are using the fact that $f^{\mathbf{B}}$ and $g^{\mathbf{B}}$ are permissible polynomials). Theorem 3.3 proves that $\mathscr{V}$ is not CSM.

For a non-unary example, let $\mathscr{V}$ be a variety of semilattices with one operator, g, axiomatized by the equations for semilattices and:

$$g(x \wedge y) \approx g(x) \wedge g(y) \qquad g^2(x) \approx g(x) \qquad g(x) \wedge x \approx x.$$

$\mathcal{V}$ is a regular variety. If $\mathbf{C} = \langle\{0, 1\}; \wedge, g\rangle$ is the algebra for which $g^{\mathbf{C}}(x) = 1$ and $0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0$ and $1 \wedge 1 = 1$, then $\mathbf{C} \in \mathcal{V}$ and $\approx_C = 1_C$. As above, Theorem 3.3 proves that $\mathcal{V}$ is not CSM. We claim that $\mathrm{typ}\{\mathcal{V}\} = \{\mathbf{5}\}$. To prove this, let $\mathbf{A}$ be any finite member of $\mathcal{V}$. $\mathbf{A}$ has a semilattice reduct, so $\mathrm{typ}\{\mathbf{A}\} \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. We will be done if we show that $\mathbf{3}, \mathbf{4} \notin \mathrm{typ}\{\mathbf{A}\}$. If this is not so, then we can find a two-element subset $\{0, 1\} \subseteq A$ and a polynomial $m(x, y, z)$ such that, on $\{0, 1\}$, m satisfies $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. We may even choose m to be commutative by forming the meet $(\wedge)$ of all polynomials obtained by permuting the variables of m. The equations of $\mathcal{V}$ imply that there must be a suitable $m(x, y, z)$ of the form $x \wedge y \wedge z \wedge a$ or $g(x) \wedge g(y) \wedge g(z) \wedge a$ for some $a \in A$. But this implies that $b(x, y) = m(x, x, y)$ is commutative, thus forcing $0 = b(0, 1) = b(1, 0) = 1$. This is a contradiction.

Every regular variety of semigroups which contains no nontrivial strongly irregular subvariety is CSM as is proved in [8]. This is the converse of Corollary 3.5 for varieties of semigroups. In general, though, a regular variety that has no strongly irregular subvarieties need not be CSM. The following counterexample is due to J. B. Nation and we reproduce it here with his kind permission.

EXAMPLE 3.9. Let $\mathcal{V}$ be the variety of groupoids axiomatized by the following equations:

$$xy \approx yx \qquad xx \approx x \qquad x(xy) \approx xy.$$

$\mathcal{V}$ is regular and it is easily seen that $\mathbf{F}_{\mathcal{V}}(x, y) = \{x, y, xy\}$, so the only possible equations witnessing strong irregularity are equivalent to $xy \approx x$. But then $x \approx xy \approx yx \approx y$, so $x \approx y$ and the only possible strongly irregular subvariety is the trivial one. Hence $\mathcal{V}$ is a regular variety with no strongly irregular subvarieties. $\mathcal{V}$ is not CSM though, as the following algebra shows. Let $\mathbf{A}$ be the algebra whose universe is $\{a, b, c\}$ and whose operation is realized as

	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

The reader can check that $\mathbf{A} \in \mathcal{V}$, but that $\mathbf{A}$ is simple of type $\mathbf{3}$. Hence $\mathcal{V}$ is not CSM by Corollary 3.4. It is also true that $1_A = \approx_A$, so Theorem 3.3 gives an alternate proof that $\mathcal{V}$ is not CSM.

We have seen in the foregoing examples that Theorem 3.3 is quite a handy way of showing that a regular variety is *not* congruence semimodular. In the other direction, using Theorem 3.3 to establish that a regular variety is CSM requires a fair bit of knowledge about the subdirectly irreducible algebras and their polynomials. If we strengthen 3.3 (3) to the condition that $\approx_A = 0_A$ for all $\mathbf{A}$ in the variety we obtain a condition that is very easy to deal with. The varieties satisfying this condition are precisely the polynomially orderable varieties. Polynomially orderable varieties are relatively easy to understand both structurally and equationally and yet they seem to nearly capture what it means for a regular variety to be CSM. We will find that these varieties satisfy the following version of the one block property.

DEFINITION 3.10. An algebra $\mathbf{A}$ has the **strong one block property** (or **strong OBP**) if it has the OBP and the unique nontrivial equivalence class of any atom of $\mathrm{Con}(\mathbf{A})$ contains exactly two elements. A class of similar algebras has the strong OBP if every member does.

$\mathbb{H}(\mathbf{A})$ has the strong OBP if and only if the inclusion map of $\mathrm{Con}(\mathbf{A})$ into the lattice $\mathrm{Eq}(A)$ of equivalence relations on A preserves covers, i.e. if $\mathrm{Con}(\mathbf{A})$ is an **isometric** sublattice of $\mathrm{Eq}(A)$. The strong OBP is more restrictive than the OBP for single algebras; any simple algebra with more than two elements shows this. However, we don't know if the strong OBP is more restrictive for varieties. Hence we ask:

PROBLEM. Is there a variety with the OBP which does not have the strong OBP?

There is no such regular variety, as the next result shows.

THEOREM 3.11. *Let $\mathcal{V}$ be a regular variety. The following conditions are equivalent.*

(1) *$\mathcal{V}$ is polynomially orderable.*
(2) *$\mathcal{V}$ has the strong OBP.*
(3) *$\mathcal{V}$ has the OBP.*
(4) *$\mathcal{V}$ is CSM and every subdirectly irreducible algebra in $\mathcal{V}$ has the strong OBP.*
(5) *Every subdirectly irreducible algebra in $\mathcal{V}$ has the strong OBP and a zero element for the monolith.*

Proof. Actually (5) $\leftrightarrow$ (1) $\rightarrow$ (2) $\rightarrow$ (3) and (2) $\rightarrow$ (4) for any variety. We will use regularity only to prove that (4) implies (5) and (3) implies (5). First, we assume (1) and deduce (2). Choose $\mathbf{A} \in \mathcal{V}$ and an atom $\alpha \in \mathrm{Con}(\mathbf{A})$. Our goal is to show that $\{a, b\} = \{c, d\}$ whenever $\alpha = \mathrm{Cg}(a, b) = \mathrm{Cg}(c, d)$. Since $(c, d) \in \mathrm{Cg}(a, b)$, it follows from Mal'cev's congruence generation theorem that $c = p(a)$ or $p(b)$ for some nonconstant unary polynomial $p(x)$, so $c \leqslant_A a$ or b. Similarly, $d \leqslant_A a$ or b, $a \leqslant_A c$

or d and $b \leqslant_A c$ or d. Let M be the set of maximal elements of the partially ordered set $\langle\{a, b, c, d\}; \leqslant_{\mathbf{A}}\rangle$. Our conclusions show that $\{a, b\} \cap M = \{c, d\} \cap M = M$, so $M \subseteq \{a, b\} \cap \{c, d\}$. If $\{a, b\} \neq \{c, d\}$, then we must have $a \equiv_\alpha b \equiv_\alpha c \equiv_\alpha d$ and that there exists $u \in \{a, b\} - M$ and $v \in \{c, d\} - M$ with $u \neq v$. The partially ordered set $\langle\{a, b, u, v\}; \leqslant_{\mathbf{A}}\rangle$ has the same set M of maximal elements. But we also have $\mathrm{Cg}(u, v) = \alpha = \mathrm{Cg}(a, b)$. Repeating the above argument with (u, v) in place of (c, d) yields the contradiction that

$$M = \{a, b\} \cap M = \{u, v\} \cap M = \emptyset$$

We conclude that $\{a, b\} = \{c, d\}$. This shows that (1) implies (2). (This result and Theorem 3.3 (3) $\to$ (5), which did not require the hypothesis of regularity, imply that in this theorem the implication (1) $\to$ (5) holds for any variety.) That (2) implies (3) is obvious while the fact that (2) implies (4) follows from Theorem 2.2. That (4) implies (5) follows from Theorem 3.3. We can finish the proof by showing that (3) implies (5) when $\mathscr{V}$ is regular and that (5) implies (1).

First, assume that $\mathscr{V}$ is a regular variety of type τ which has the OBP. $\mathscr{V}$ is CSM, so every subdirectly irreducible algebra in $\mathscr{V}$ has a zero element for the monolith. We only need to prove that each subdirectly irreducible algebra $\mathbf{B} \in \mathscr{V}$ has the strong OBP. We will assume otherwise and argue to a contradiction. Let $\mu \in \mathrm{Con}(\mathbf{B})$ be the monolith of $\mathbf{B}$ and let $z \in B$ be the zero element for μ. Since $\mathbf{B}$ does not have the strong OBP we can find $a \neq b \in B$ such that $m = \mathrm{Cg}(a, z) = \mathrm{Cg}(b, z)$. From what we know of the structure of a minimal congruence in a regular, CSM variety, we must have $z \leqslant_B a \approx_B b \leqslant_B c$ for any $c \in B - \{z\}$. Consider $\mathbf{C} = (\mathbf{B} \times \mathbf{S}_\tau)/\mathrm{Cg}(z0, z1)$. Since $a \approx_B b$ we have $\mathrm{Cg}^{\mathbf{C}}(a0, a1) = \mathrm{Cg}^{\mathbf{C}}(b0, b1) \succ 0_C$. Using the OBP in $\mathbf{C}$, we get that $(a0, b0) \in \mathrm{Cg}^{\mathbf{C}}(\{a0, b0, a1, b1\}) = \mathrm{Cg}^{\mathbf{C}}(a0, a1)$. But $(xy, uv) \in \mathrm{Cg}^{\mathbf{C}}(a0, a1)$ implies that $x = u$. This forces $a = b$ which is false. Thus, (3) implies (5).

Now, assume that (5) holds and that (1) fails. Choose $\mathbf{D} \in \mathscr{V}$ which has elements u, v and unary polynomials p and q, each a composition of permissible polynomials, such that $p(u) = v$ and $q(v) = u$. Factoring out by a congruence which is maximal with respect to the property of not containing (u, v) we obtain a subdirectly irreducible algebra $\mathbf{E}$ with elements $\bar{u}$ and $\bar{v}$ such that $\mathrm{Cg}(\bar{u}, \bar{v})$ is the monolith of $\mathbf{E}$. Further, $\mathbf{E}$ has polynomials $\bar{p}$ and $\bar{q}$ which are compositions of permissible polynomials such that $\bar{p}(\bar{u}) = \bar{v}$ and $\bar{q}(\bar{v}) = \bar{u}$, so $\bar{u} \approx_E \bar{v}$. But condition (5) implies that the monolith of $\mathbf{E}$ is just the equivalence relation generated by $(\bar{u}, \bar{v})$. The unique nontrivial block of the monolith can only be $\{\bar{u}, \bar{v}\}$. It is impossible for this block to have a zero element if $\bar{u} \approx_E \bar{v}$, so we have a contradiction. □

Polynomially orderable varieties seem to be the simplest kind of regular, CSM variety, so we will examine them a little closer. The next two theorems provide some extra information about regular, polynomially orderable varieties.

THEOREM 3.12. *Assume that $\mathcal{V}$ is a nontrivial, regular, polynomially orderable variety whose set of fundamental operations has cardinality κ and whose subset of unary fundamental operations has cardinality λ. The following are true.*

(1) *$\mathcal{V}$ has $\leq 2^\kappa$ non-isomorphic simple algebras, each of cardinality 2 and each generating distinct minimal subvarieties of $\mathcal{V}$. There are $\leq 2^\lambda$ non-isomorphic abelian simple algebras and each one is essentially unary.*

(2) *Every finite $\mathbf{A} \in \mathcal{V}$ has a two-element subuniverse equal to a congruence block.*

(3) *$\mathcal{V}$ is* **precomplete** *(i.e. has exactly one minimal subvariety) if and only if $\mathcal{V}$ is idempotent.*

(4) $\operatorname{typ}\{\mathcal{V}\} = \{\mathbf{0}\}$ *if and only if $\mathcal{V}$ is unary.* $\operatorname{typ}\{\mathcal{V}\} = \{\mathbf{5}\}$ *if and only if $\mathcal{V} \models f(x, x, \ldots, x) \approx x$ for some fundamental operation f or arity ≥ 2.*

Proof. If $\mathbf{S}$ is a simple algebra in $\mathcal{V}$, then $|S| = 2$ since $\mathcal{V}$ has the strong OBP. By changing to an isomorphic algebra we may assume that $S = \{0, 1\}$ where 1 is not $\ll_S 0$. If h is an n-ary fundamental operation, then the polynomial $h^{\mathbf{S}}(s_0, \ldots, s_{i-1}, x, s_{i-1}, \ldots, s_{n-1})$, $s_j \in S$, is permissible and cannot map 0 to 1. It follows that either (I) $h^{\mathbf{S}}(S^n) = \{0\}$ or else (II):

$$h^{\mathbf{S}}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i = 1 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

If 0 is an absorbing element for one fundamental operation, then it must be a zero element for $\mathbf{S}$ or else we contradict the fact that 1 is not $\ll_A 0$. Hence each simple algebra is determined up to isomorphism by specifying the subset of the fundamental operations which depend on at least one variable. This shows that there are $\leq 2^\kappa$ non-isomorphic simple algebras in $\mathcal{V}$. If a fundamental operation of arity $n > 1$ depends on a variable, then it realizes the essentially n-ary semilattice operation on S. This forces $\mathbf{S}$ to be nonabelian. Hence, if $\mathbf{S}$ is abelian, every fundamental operation of arity > 1 must be independent of all variables. Such an operation falls under case (I) from above. The unary operations of $\mathbf{S}$ may be of the type described in case (I) or case (II), so there can be at most 2^λ such algebras and each is essentially unary.

To show that every simple algebra generates a minimal subvariety notice that each simple algebra which has a nonconstant fundamental operation of arity > 1 generates a variety equivalent to the variety of semilattices or to the variety of semilattices with a zero element; such varieties have no nontrivial subvarieties. In the case that every nonconstant operation is unary, every simple algebra generates a variety equivalent to the variety of sets or the variety of pointed sets; again these are minimal varieties. If $\mathbf{A}$ and $\mathbf{B}$ are non-isomorphic simple algebras in $\mathcal{V}$, then there is a fundamental operation which depends on a variable in one of these algebras but does not depend on any variable in the other. This can be expressed equationally, so $\mathbb{V}(\mathbf{A}) \neq \mathbb{V}(\mathbf{B})$. This establishes (1).

To prove (2), choose an element $0 \in A$ which is minimal under the partial ordering $\preccurlyeq_A$. (If $\mathcal{V}$ is not unary, then we can find a binary polynomial $p(x, y) = t^{\mathbf{A}}(x, y, \bar{c})$ where $t(x, y, \bar{z})$ depends on all of its variables in $\mathcal{V}$. Then for all $a, b \in A$ we get $p(a, b) \preccurlyeq_A a, b$. Hence the poset $\langle A; \preccurlyeq_A \rangle$ is **downward-directed**. In this case the choice of 0 is unique.) Obviously 0 is a zero element for $\mathbf{A}$. Now let c be any element which is minimal in $A - \{0\}$. The set $\{0, c\}$ is preserved by all permissible polynomials, so it is the unique congruence class of a (minimal) congruence on $\mathbf{A}$. It is also the subuniverse of a simple subalgebra of $\mathbf{A}$.

To prove (3), assume that $\mathcal{V}$ has only one minimal subvariety. Since $\mathcal{V}$ is regular the minimal subvariety can only be $\mathbb{V}(\mathbf{S}_\tau)$. Thus every subvariety of $\mathcal{V}$ is regular. We need to show that every term operation (equivalently, every fundamental operation) is idempotent. If $\mathcal{V}$ is not idempotent, then we can find a $\mathbf{B} \in \mathcal{V}$, an element $b \in B$ and a fundamental operation g such that $g^{\mathbf{B}}(b, \ldots, b) \neq b$. Factoring by a congruence maximal with respect to not containing $(g^{\mathbf{B}}(b, \ldots, b), b)$ if necessary, we may assume that $\mathbf{B}$ is subdirectly irreducible algebra and that the equivalence relation ν generated by $(g^{\mathbf{B}}(b, \ldots, b), b)$ is the monolith of $\mathbf{B}$. From Theorem 3.3 it follows that the nontrivial ν-class contains a zero element which can't be b so $g^{\mathbf{B}}(b, \ldots, b)$ is a zero element of $\mathbf{B}$. In particular, $g^{\mathbf{B}}(b, \ldots, b)$ is a one-element subuniverse of $\mathbf{B}$. Therefore, the congruence class $\{g^{\mathbf{B}}(b, \ldots, b), b\}$ is also a subuniverse; call the corresponding two-element subalgebra $\mathbf{C}$. Since $g^{\mathbf{C}}$ is constant the variety generated by $\mathbf{C}$ satisfies an (irregular) equation of the form $g(x_0, \ldots, x_m) \approx g(y_0, \ldots, y_m)$ where the x_i's are distinct from the y_j's. This contradicts our earlier observation that every subvariety of $\mathcal{V}$ is regular. This proves that every fundamental operation is idempotent. Conversely, assume that $\mathcal{V}$ is idempotent. No term operation is constant on any member of $\mathcal{V}$. A quick examination of the simple algebras described in the proof of part (1) shows that $\mathcal{V}$ contains exactly one nontrivial simple algebra: $\mathbf{S}_\tau$ where τ is the type of $\mathcal{V}$. Every minimal subvariety of $\mathcal{V}$ is generated by a nontrivial simple algebra, so $\mathbb{V}(\mathbf{S}_\tau)$ is the unique minimal subvariety of $\mathcal{V}$.

For the first part of (4), notice that if $\mathcal{V}$ is unary, then $\mathcal{V}$ is strongly abelian and CSM. It follows that $\mathrm{typ}\{\mathcal{V}\} = \{\mathbf{0}\}$. Conversely, if $\mathcal{V}$ is not unary, then $\mathbf{5} = \mathrm{typ}\{\mathbf{S}_\tau\} \in \mathrm{typ}\{\mathcal{V}\}$. For the second part of (4), we clearly have $\mathrm{typ}\{\mathcal{V}\} = \{\mathbf{5}\}$ if and only if $\mathbf{0} \notin \mathrm{typ}\{\mathcal{V}\}$. If a finite subdirectly irreducible algebra has a monolith μ of type $\mathbf{0}$, then the unique nontrivial μ-class is the universe of a two-element subalgebra which is essentially unary. Conversely, any essentially unary simple algebra in $\mathcal{V}$ witnesses the fact that $\mathbf{0} \in \mathrm{typ}\{\mathcal{V}\}$. Hence $\mathrm{typ}\{\mathcal{V}\} = \{\mathbf{5}\}$ if and only if $\mathcal{V}$ contains no essentially unary simple algebra. If $\mathcal{V} \models f(x, x, \ldots, x) \approx x$ for some fundamental operation f of arity ≥ 2, then f depends on all variables in every algebra in $\mathcal{V}$. This condition is sufficient to prove that $\mathcal{V}$ contains no essentially unary simple algebras. Conversely, suppose that $\mathcal{V}$ satisfies no such equation. That is, assume that for each fundamental operation f of arity ≥ 2 we can find a $\mathbf{D} \in \mathcal{V}$ and an element $d \in D$ such that $f^{\mathbf{D}}(d, \ldots, d) \neq d$. As the argument of the previous paragraph shows we can assume that $\mathbf{D}$ is simple. If F is the set of operation symbols, then for each $f \in F$ of arity $n \geq 2$ we can find a simple algebra $\mathbf{D}_f = \langle \{0, 1\}; F \rangle$ such that $f^{\mathbf{D}_f}(D_f^n) = \{0\}$. In each such algebra $g^{\mathbf{D}_f}(0) = 0$ for any unary fundamental operation, g. Let $\mathbf{E} = \prod_{f \in F} \mathbf{D}_f$. For any $f \in F$ of arity ≥ 2 the range of $f^{\mathbf{D}}$ is contained in the set $X = E - \{(1, 1, \ldots, 1)\}$. Further, X is closed under all the unary fundamental operations. Hence X is a block of a congruence θ on $\mathbf{E}$ and $\mathbf{E}/\theta$ is a simple algebra for which every basic operation of arity ≥ 2 depends on no variables. $\mathbf{E}$ is essentially unary, so $\mathbf{0} \in \mathrm{typ}\{\mathcal{V}\}$. This establishes (4). □

The next theorem characterizes generator classes for locally finite, polynomially orderable varieties. Later, in Theorems 3.17 and 3.18, we characterize certain polynomially orderable varieties equationally.

THEOREM 3.13. *If $\mathcal{V} = \mathbb{V}(\mathcal{K})$ is a locally finite variety of type τ, then $\mathcal{V}$ is a regular, polynomially orderable variety if and only if*

(1) *Each algebra in $\mathcal{K}$ can be partially ordered in such a way that for each fundamental operation f we have*

$$\mathcal{K} \models f(x_0, \ldots, x_n) \leq x_0, \ldots, x_n.$$

(2) $\mathbf{S}_\tau \in \mathbb{S}(\mathcal{K})$.

Proof. Assume that $\mathcal{V}$ is a regular, polynomially orderable variety. For each $\mathbf{A} \in \mathcal{K}$ choose $\leq \; = \; \ll_A$ and observe that $p_i(x) = f^{\mathbf{A}}(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is permissible. Thus, $p_i(a_i) \leq a_i$ and so condition (1) holds. To prove condition (2), let $\mathbf{F}$ be the 2-generated, relatively free algebra in $\mathcal{V}$. There is an onto homomorphism $\phi: \mathbf{F} \to \mathbf{S}_\tau$. Let $1'$ be an element of $\phi^{-1}(1)$ which is minimal under $\ll_F$ and let $0'$ be an element of $\phi^{-1}(0)$ which is minimal under $\ll_F$. It is straightforward to show that $\{0', 1'\} \subseteq F$ is a subuniverse for an algebra isomorphic to $\mathbf{S}_\tau$. Since $\mathbf{S}_\tau$ embeds into $\mathbf{F} \in \mathbb{SP}(\mathcal{K})$ and $\mathbf{S}_\tau$ is simple, it must be that $\mathbf{S}_\tau \in \mathbb{S}(\mathcal{K})$.

Now assume (1) and (2). $\mathcal{V}$ is regular by (2). To finish this proof we will show that $\mathcal{V}$ has the OBP and then invoke Theorem 3.11. By Theorem 2.4, it suffices to prove that the finite algebras in $\mathcal{V}$ have the OBP. Let $\mathbf{B}$ be a finite algebra in $\mathcal{V}$ and assume that $0_B \prec \beta$ in $\mathrm{Con}(\mathbf{B})$. We want to prove that β has exactly one nontrivial block. Choose a finite, relatively free algebra $\mathbf{G}$ in $\mathscr{V}$ which has a homomorphism onto $\mathbf{B}$. Suppose that $\alpha \prec \gamma$ in $\mathrm{Con}(\mathbf{G})$, $\mathbf{G}/\alpha \cong \mathbf{B}$ and γ/α corresponds to β under this isomorphism. We must show that exactly one γ-class is different from an α-class. Now G inherits an ordering from the orderings of each member of $\mathscr{K}$: as $\mathbf{G}$ is embeddable into a product of members of $\mathscr{K}$, one can restrict this product ordering to G. We will use the same symbol, $\leq$, to denote this ordering on G. Now let $u \in G$ be an element which is minimal under $\leq$ for the property that there exists a $w \in G$ such that $(u, w) \in \gamma - \alpha$. Let $v \in G$ be an element which is minimal under $\leq$ for the property that $(u, v) \in \gamma - \alpha$. Clearly, for each fundamental operation f we have $\mathbf{G} \models f^{\mathbf{G}}(g_0, \ldots, g_n) \leq g_0, \ldots, g_n$. Using this and induction on the complexity of a polynomial one can prove that if p is any unary polynomial of $\mathbf{G}$ and $p(u) \neq u$ or $p(v) \neq v$, then $(p(u), p(v)) \in \alpha$. This means that $\gamma = \alpha \vee \mathrm{Cg}(u, v)$ is equal to the equivalence relation generated by $\alpha \cup \{u, v\}$. We find that the only class of γ that differs from an α-class is the class containing u and v. This finishes the proof. □

Condition (2) of Theorem 3.13 is only necessary to ensure that the generated variety is regular. In fact, if $\mathscr{K}$ satisfies 3.13 (1), then $\mathrm{reg}\,\mathscr{V}$ is a regular, polynomially orderable variety as one sees by applying the theorem to $\mathbb{V}(\mathscr{K} \cup \{\mathbf{S}_\tau\})$.

EXAMPLE 3.14. The variety of sets and the variety of semilattices are examples of regular, polynomially orderable varieties. Another example is the variety of **directoids** introduced in [7]. The variety of directoids is the variety of type $\langle 2 \rangle$ which is axiomatized by the equations

$$x \cdot x \approx x, \qquad (x \cdot y) \cdot x \approx x \cdot y, \qquad y \cdot (x \cdot y) \approx x \cdot y,$$
$$x \cdot ((x \cdot y) \cdot z) \approx (x \cdot y) \cdot z.$$

Ježek and Quackenbush prove that a groupoid $\mathbf{A}$ is a directoid if and only if A can be partially ordered, say by $\leq$, such that (i) $x, y \leq x \cdot y$ holds for all $x, y \in A$ and (ii) whenever $x \leq y$ one has $x \cdot y = y \cdot x = y$. Any two comparable elements of a directoid form a subuniverse isomorphic to $\mathbf{S}_{\langle 2 \rangle}$, so the variety of directoids is regular. By reversing the order suggested in [7] to correspond to the downward-directed ordering that we have been considering, it becomes clear that the variety of directoids is polynomially orderable. This variety is idempotent, so it has exactly one minimal subvariety, $\mathbb{V}(\mathbf{S}_{\langle 2 \rangle})$. (This is proved in [7]. In [7] it is further shown that in the lattice of varieties of directoids $\mathbb{V}(\mathbf{S}_{\langle 2 \rangle})$ has exactly one cover and that this cover has exactly four covers.)

EXAMPLE 3.15. A variety of combinatorial inverse semigroups is any variety of type $\langle 2, 1\rangle$ which, for some n, satisfies the equations

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z), \qquad x^{n+1} \approx x^n, \qquad x \cdot x' \cdot x \approx x,$$
$$(x')' = x, \qquad (x \cdot y)' \approx y' \cdot x', \qquad x \cdot x' \cdot y \cdot y' \approx y \cdot y' \cdot x \cdot x'.$$

In [8], P. Jones proved that any variety of combinatorial inverse semigroups is CSM. When $n = 1$ these equations imply $x' \approx x$ and the variety defined, $\mathcal{S}$, is equivalent to the variety of semilattices as one can see by setting $y = x'$ in the last equation. This variety is polynomially orderable and it is known that $\mathcal{S}$ is the only equationally complete variety of combinatorial inverse semigroups. In Corollary XII.4.14 of [11], it is shown that the variety $\mathcal{B}$ defined by the above equations with $n = 2$ and also the equation $(y \cdot x \cdot y')^2 \approx y \cdot x \cdot y'$ is a (necessarily join irreducible) cover of $\mathcal{S}$. $\mathcal{B}$ has no bound on the size of its simple algebras. To see this, let I be any set and let $A = I \times I \cup \{0\}$. Define $0' = 0$ and $(x, y)' = (y, x)$. Define $(x, y) \cdot (u, v) = (x, v)$ if $y = u$ and define $(x, y) \cdot (u, v) = 0$ if $y \neq u$. Define $0 \cdot (x, y) = (x, y) \cdot 0 = 0$. Then $\mathbf{A} = \langle A; \cdot, '\rangle \in \mathcal{B}$ and it is a rather easy exercise to prove that $\mathbf{A}$ is simple. This proves our claim. Each simple algebra of more than 2 elements is not a member of $\mathcal{S}$ and so must generate the non-minimal variety $\mathcal{B}$. Neither basic operation is idempotent in $\mathcal{B}$ even though $\mathcal{B}$ is precomplete and $\operatorname{typ}\{\mathcal{B}\} = \{\mathbf{5}\}$. Thus, $\mathcal{B}$ is a regular, CSM variety which is not polynomially orderable and, in fact, fails every conclusion of Theorem 3.12.

From Theorem 3.3 we see that every subdirectly irreducible algebra in a regular, CSM variety has the OBP. If, moreover, each subdirectly irreducible algebra has the strong OPB, then the variety is polynomially orderable. For semigroups, the regular, CSM varieties and polynomially orderable varieties seem to be close equationally as well. For example, compare the equational characterization of regular, CSM varieties of semigroups given in [8] with the characterization of regular, polynomially orderable varieties of semigroups given in [3]:

THEOREM 3.16. [8] *A regular variety of semigroups is CSM if and only if for some n the variety satisfies*:

$$x^n \approx x^{n+1} \quad \text{and} \quad (x^n y^n)^n \approx (y^n x^n)^n. \qquad \square$$

THEOREM 3.17. [3] *A regular variety of semigroups consists of J-trivial semigroups if and only if for some n the variety satisfies*:

$$x^n \approx x^{n+1} \quad \text{and} \quad (xy)^n \approx (yx)^n. \qquad \square$$

To make the comparison we mentioned it is necessary to prove that a regular variety of semigroups consists of J-trivial semigroups if and only if it is polynomially orderable. (The J-relation on a semigroup **S** is the set of pairs $(a, b) \in S^2$ where a and b generate the same 2-sided ideal. **S** is **J-trivial** if $J = 0_A$. See Section I.6 of [11] for more details.) Notice that the image of $a \in S$ under any permissible polynomial is a member of the 2-sided ideal, I_a, which is generated by a. The set of all such images is all of I_a. (This is the only place where we use regularity; we use the fact that multiplication is permissible.) Now, $a \approx_S b$ if and only if $I_a = I_b$, so $\approx_S = 0_S$ if and only if **S** is J-trivial.

The next result examines regular, polynomially orderable, unary varieties. For this result, an element p of a monoid M has **finite order** if there exists a positive integer n such that $p^n = p^{2n}$. If every element of M has finite order, then we will write this as $M \models x^\omega \approx x^{2\omega}$ to mean that for each $p \in M$ there is some $n < \omega$, possibly depending on p, such that $p^n = p^{2n}$.

THEOREM 3.18. *Suppose that $\mathcal{U}$ is a regular, unary variety and that M is the monoid of unary term operations. The following conditions are equivalent*:

(1) *$\mathcal{U}$ is polynomially orderable and every element of M has finite order.*

(2) *$M \models x^\omega \approx x^{\omega+1}$ and $(xy)^\omega \approx y(xy)^\omega$.*

If, in (1), *$M \models x^n \approx x^{2n}$ then, in* (2), *$M \models x^n \approx x^{n+1}$ and $(xy)^n \approx y(xy)^n$.*

Proof. In a regular, unary variety the permissible polynomials of any algebra are precisely the interpretations of the unary terms. Choose $p \in M$ and n such that $M \models p^n \approx p^{2n}$. If $\mathbf{A} \in \mathcal{U}$ and $a \in A$ let $b = p^n(a)$.

$$b = p^n(a) = p^{2n}(a) = p^n(b) = p^{n-1}(p(b)),$$

so $b \leqslant_A p(b)$ and, of course, $p(b) \leqslant_A b$, so $b \approx_A p(b)$. If $\mathcal{U}$ is polynomially orderable, then $p^n(a) = b = p(b) = p^{n+1}(a)$. Since a was arbitrary $\mathbf{A} \models p^n \approx p^{n+1}$ and therefore $M \models x^\omega \approx x^{\omega+1}$. Now suppose that $p, q \in M$. From what we've shown there is an m such that $M \models (pq)^m \approx (pq)^{m+1}$. If $c \in A$, then let $d = (pq)^m(c)$. We have $p(q(d)) = d$ and this give us $d \leqslant_A q(d)$ while we clearly have $q(d) \leqslant_A d$. As before, this yields $d = q(d)$ or $(pq)^m(c) = q(pq)^m(c)$. The choice of c was arbitrary, so $\mathbf{A} \models (pq)^m \approx q(pq)^m$. Since p and q were arbitrary $M \models (xy)^\omega \approx y(xy)^\omega$. This shows that (1) implies (2).

The fact that $M \models x^\omega \approx x^{\omega+1}$ implies that every element of M is finite order. To show that (2) implies (1) we only need to show that $\mathcal{U}$ is polynomially orderable. Assume that $\mathbf{B} \in \mathcal{U}$ and $e, f \in B$ satisfy $e \approx_B f$. This means that we can find permissible polynomials $r, s \in M$ such that $r(e) = f$ and $s(f) = e$. Now there is some k such that $(rs)^k = s(rs)^k$. Thus, $f = (rs)^k(f) = s(rs)^k(f) = e$. That is, $e \approx_B f$ implies

$e = f$ whenever $e, f \in B$ and $\mathbf{B} \in \mathcal{U}$. This is just what it means for $\mathcal{U}$ to be polynomially orderable.

It is clear that if there is a uniform bound on the order of the elements in M, say $M \models x^n \approx x^{2n}$, then the same arguments we have used lead to $M \models x^n \approx x^{n+1}$ and $(xy)^n \approx y(xy)^n$. $\square$

Every locally finite, unary variety has a finite monoid of unary term operations. Every finite monoid satisfies an equation of the form $x^n \approx x^{2n}$ for some n, so Theorem 3.18 contains a characterization of all locally finite, regular, unary, polynomially orderable varieties. A locally finite, regular, unary variety is a polynomially orderable variety if and only if it satisfies hyper-identities of the form $x^n \approx x^{n+1}$ and $(xy)^n \approx y(xy)^n$ for some n (see [13] for the definition of **hyper-identity**). Perhaps this should be phrased as: A locally finite, regular, unary variety is a polynomially orderable variety if and only if it satisfies the **hyper-pseudoidentities** $x^\omega \approx x^{\omega+1}$ and $(xy)^\omega \approx y(xy)^\omega$. (See [13] for the definition of **pseudoidentity**.)

It is nearly true that hyper-pseudoidentities similar to Theorem 3.18 characterize all locally finite, regular, polynomially orderable varieties. For example, we might write $t_{\bar{y}}(x)$ for $t(x, \bar{y})$ and juxtaposition, st, for $s_{\bar{z}}(t_{\bar{y}}(x))$. Then the arguments of Theorem 3.18 almost show that any locally finite, regular, polynomially orderable variety satisfies $t^\omega \approx t^{\omega+1}$ and $(st)^\omega \approx t(st)^\omega$ and that, conversely, any locally finite, regular variety satisfying these hyper-pseudoidentities is polynomially orderable. The difficulty is this: some n-variable terms do not involve all n variables. For example, the ith projection, $t_i(x_0, \ldots, x_n) \approx x_i$, involves only x_i. Hence, if $\mathbf{A} \in \mathcal{V}$, the polynomial $p(x) = t_i^{\mathbf{A}}(x, a_1, \ldots, a_n)$ where each $a_j \in A$ is not permissible when $i > 0$. The sentences $t^\omega \approx t^{\omega+1}$ and $(st)^\omega \approx t(st)^\omega$ characterize locally finite, regular, polynomially orderable varieties only if these sentences are quantified over the terms that involve the first variable.

Acknowledgements

The authors would like to acknowledge R. Freese, W. Lampe, R. McKenzie and J. B. Nation for their encouragement and help. We are also grateful to the referees for their meticulous work and thoughtful suggestions.

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