COLLAPSING PERMUTATION GROUPS

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Abstract. It is shown in [3] that any nonregular quasiprimitive permutation group is collapsing. In this paper we describe a wider class of collapsing permutation groups.

Dedicated to Ivo Rosenberg on his 65th birthday

1. Introduction

Let $\mathbf{A} = (A; F)$ be a finite algebra. The unary part of the clone of $\mathbf{A}$, $M = \mathrm{Clo}_1\,\mathbf{A}$, is a transformation monoid on the set A. In the lattice of clones on A, the collection of clones whose unary part is M forms an interval. It has long been known that if $|A| \geq 3$ and M consists of all the constant operations and the identity function, then this interval has only one element. In other words, there is only one algebra up to term equivalence whose base set is A and whose unary term operations are exactly the constant operations together with the identity function: it is the unary algebra $(A; M)$.

This result is extended by P. P. Pálfy in [2]. Pálfy proves that if $\mathbf{A} = (A; F)$ is an algebra with $|A| \geq 3$ and the transformation monoid $M = \mathrm{Clo}_1\,\mathbf{A}$ consists of all constant operations on A together with a group of permutations, then $(A; F)$ is term equivalent to the unary algebra $(A; M)$ or to a vector space. Thus, if M consists of all constant operations together with a group of permutations where the group is not the affine group of a vector space on A, then the only algebra up to term equivalence whose base set is A and whose unary term operations are exactly the operations in M is $(A; M)$.

These results motivate the following definition (cf. [3]): a transformation monoid M on A is **collapsing** if the only algebra up to term equivalence whose base set is A and whose unary term operations are exactly the operations in M is $(A; M)$. In this

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paper we will describe new results about collapsing monoids that are permutation groups.

Let Γ be a permutation group on A. In [3] it is shown that if Γ is a nonregular quasiprimitive group, then it is collapsing. The requirement that Γ is nonregular means that some $\gamma \in \Gamma \setminus \{1\}$ has a fixed point. That Γ is quasiprimitive means that every nontrivial normal subgroup of Γ acts transitively on A. This is equivalent to the condition that for any nontrivial normal subgroup N of Γ it is the case that $N\Gamma_a = \Gamma$ when Γ_a is a 1-point stabilizer. In [5] the reader will find an O'Nan–Scott type theorem classifying quasiprimitive groups.

In this paper we describe a wider class of collapsing permutations groups.

Theorem 1.1. *Let Γ be a nonregular transitive permutation group acting on a finite set A. If for every normal subgroup N of Γ either $N\Gamma_a = \Gamma$ or $N \cap \Gamma_a = \{\mathrm{id}\}$, then Γ is collapsing.*

The condition in Theorem 1.1 says that every normal subgroup of Γ acts transitively or semiregularly on A.

In the last section of the paper we describe some examples of collapsing permutation groups which do not satisfy the condition described in Theorem 1.1. The arguments we use to show that these examples are collapsing suggest that a full characterization of collapsing permutation groups will require a study of 'geometries' associated to finite groups.

2. G-algebras

Let G be a group. An algebra $\mathbf{G}$ is called a G**-algebra** if the universe of $\mathbf{G}$ is G and $\mathrm{Clo}_1\,\mathbf{G}$ coincides with the group L_G of left translations of G. It is easy to see that an algebra $\mathbf{G}$ with universe G is a G-algebra if and only if the following two conditions hold for $\mathbf{G}$:

(1) all permutations in L_G are term operations of $\mathbf{G}$, and
(2) all permutations in the group R_G of right translations of G are automorphisms of $\mathbf{G}$.

An operation f defined on the set G will be called a G**-operation** if all permutations in R_G are automorphisms of the algebra $(G; f)$. Clearly, an operation is a term operation of a G-algebra exactly when it is a G-operation.

It follows from the definition that every $(n+1)$-ary G-operation $f(x, y_1, \dots, y_n)$ is determined by the associated n-ary operation $\widetilde{f}(y_1, \dots, y_n) = f(1, y_1, \dots, y_n)$ where 1 is the unit element of the group G; namely, for any $x \in G$ we have

$$\begin{aligned} f(x, y_1, \dots, y_n) &= f(1 \cdot x, (y_1x^{-1}) \cdot x, \dots, (y_nx^{-1}) \cdot x) \\ &= f(1, y_1x^{-1}, \dots, y_nx^{-1}) \cdot x \\ &= \widetilde{f}(y_1x^{-1}, \dots, y_nx^{-1}) \cdot x. \end{aligned}$$

Conversely, every n-ary operation $\widetilde{f}(y_1, \ldots, y_n)$ on G gives rise to an $(n+1)$-ary G-operation $f(x, y_1, \ldots, y_n)$ via the definition

$$f(x, y_1, \ldots, y_n) = \widetilde{f}(y_1 x^{-1}, \ldots, y_n x^{-1}) \cdot x. \tag{2.1}$$

With this definition, $\widetilde{f}(y_1, \ldots, y_n)$ agrees with $f(1, y_1, \ldots, y_n)$.

We will now explain how the claim in Theorem 1.1 can be translated into a claim about G-algebras. If G is a group and H is a subgroup of G, then α_H will denote the equivalence relation on G whose blocks are the left cosets gH ($g \in G$) of H. It is easy to see that the congruences of the unary algebra $(G; L_G)$ are exactly the equivalence relations α_H where H runs over all subgroups of L_G. For each subgroup H of G the quotient algebra $(G; L_G)/\alpha_H$ is $(G/\alpha_H; L_G[H])$ where $L_G[H]$ denotes the group of left multiplications by elements of G, as they act on the left cosets of H. Clearly, $L_G[H]$ is a transitive permutation group on G/α_H; moreover, the permutation group $L_G[H]$ is regular if and only if H is a normal subgroup of G. It is also well known and easy to check that the natural group homomorphism $L_G \to L_G[H]$ is one-to-one if and only if H contains no nontrivial normal subgroup of G; or equivalently, if and only if the intersection of the conjugates $H^g = g^{-1}Hg$ of H is the one-element group. In this case the subgroup H of G will be called **core-free**.

It is well known that every transitive permutation group can be represented in the form $L_G[H]$ where G is a group and H is a core-free subgroup of G. Therefore every algebra whose unary term operations form a transitive permutation group can be thought of as an algebra $\mathbf{Q}$ which is defined on the set G/α_H and satisfies $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$ for such a pair G and H. The next lemma describes the relationship between these algebras $\mathbf{Q}$ and those G-algebras which admit α_H as a congruence.

Lemma 2.1. *Let G be a group and let H be a core-free subgroup of G.*

1. *For every G-algebra $\mathbf{G}$ which admits α_H as a congruence the quotient algebra $\mathbf{G}/\alpha_H$ generates the same variety as $\mathbf{G}$. Consequently the natural homomorphism $\mathrm{Clo}\,\mathbf{G} \to \mathrm{Clo}\,(\mathbf{G}/\alpha_H)$ between their clones is an isomorphism. We have $\mathrm{Clo}_1\,\mathbf{G} = L_G$ and $\mathrm{Clo}_1\,(\mathbf{G}/\alpha_H) = L_G[H]$.*
2. *Conversely, if $\mathbf{Q}$ is an algebra with universe G/α_H such that $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$ then there exists a G-algebra $\mathbf{G}$ such that $\mathbf{Q} = \mathbf{G}/\alpha_H$.*
3. *The mapping*
$$\mathrm{Clo}\,\mathbf{G} \mapsto \mathrm{Clo}\,(\mathbf{G}/\alpha_H)$$
defines a lattice isomorphism between the clones of those G-algebras which admit α_H as a congruence, and the clones of those algebras $\mathbf{Q}$ (on G/α_H) which satisfy $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$.

Proof. (1) We start the proof with a claim which, for later use, will be stated for arbitrary subgroups of G, not only for core-free subgroups.

Claim 2.2. *Let G be a group, let $\overline{H}$ be an arbitrary subgroup of G, and let N denote the intersection of the conjugates of $\overline{H}$. For any G-algebra $\mathbf{G}$ which admits $\alpha_{\overline{H}}$ as a congruence, α_N is a congruence of $\mathbf{G}$, and $\mathbf{G}/\alpha_N$ is isomorphic to a subdirect power of $\mathbf{G}/\alpha_{\overline{H}}$.*

To prove the claim let $\mathbf{G}$ be a G-algebra which admits $\alpha_{\overline{H}}$ as a congruence. Since the elements of the permutation group R_G are automorphisms of $\mathbf{A}$, therefore $\alpha_{\overline{H}^g}$ is a congruence of $\mathbf{A}$ for every conjugate $\overline{H}^g = g^{-1}\overline{H}g$ of $\overline{H}$. The intersection of all these congruences is obviously α_N, and the quotients $\mathbf{G}/\alpha_{\overline{H}^g}$ are all isomorphic to $\mathbf{G}/\alpha_{\overline{H}}$. This concludes the proof of the claim.

The statement in the first sentence of (1) is an immediate consequence of this claim. The rest of (1) is clear.

To verify (2) let us consider an algebra $\mathbf{Q}$ with $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$. There is a 1-generated free algebra $\mathbf{F}$ in the variety generated by $\mathbf{Q}$ that has base set $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$, and the operations are defined in the obvious way. Now one can see that this free algebra is an $L_G[H]$-algebra, and the kernel of the natural homomorphism $\mathbf{F} \to \mathbf{Q}$ is α_U where U is the stabilizer subgroup of H in $L_G[H]$. Under the assumptions of the lemma the group homomorphism $G \to L_G[H]$ assigning to each $g \in G$ the permutation 'multiplication by g' (as it acts on the left cosets of H) is an isomorphism; furthermore, it is clear that under this isomorphism the stabilizer U corresponds exactly to the subgroup H of G. Therefore, via renaming the elements of $\mathbf{F}$ we can get a G-algebra $\mathbf{G}$ as required in (2).

In (3) the only nontrivial claim is that the mapping described is a bijection. (2) shows that it is surjective. To prove that it is also injective let $\mathbf{Q}$ be an algebra (on G/α_H) which satisfies $\mathrm{Clo}_1\,\mathbf{Q} = L_G[H]$, and let $\mathbf{G}$ be a G-algebra which admits α_H as a congruence and satisfies $\mathbf{G}/\alpha_H = \mathbf{Q}$. We have seen in (1) that $\mathbf{G}$ must generate the same variety as $\mathbf{Q}$. However, since $\mathbf{G}$ is a G-algebra, it is the one-generated free algebra in the variety it generates. Therefore $\mathbf{G}$ is uniquely determined by $\mathbf{Q}$. □

For our purposes the following special case of Lemma 2.1 will be useful.

Corollary 2.3. *Let G be a group, and let H be a core-free subgroup of G. The permutation group $L_G[H]$ is collapsing if and only if $(G; L_G)$ is the only G-algebra, up to term equivalence, which admits α_H as a congruence.*

For a group G and a nontrivial subgroup H in G we will say that $\langle G, H\rangle$ is a **collapsing pair** if $(G; L_G)$ is the only G-algebra, up to term equivalence, which admits α_H as a congruence.

3. Proof of Theorem 1.1

From now on we will focus on finite groups. Since every transitive permutation group on a finite set can be represented in the form $L_G[H]$ for some finite group G and some core-free subgroup H of G, Corollary 2.3 shows that Theorem 1.1 is equivalent to the following statement.

Theorem 3.1. *Let G be a finite group, and let H be a nontrivial proper subgroup of G. If for every normal subgroup N of G either $NH = G$ or $N \cap H = \{1\}$, then the pair $\langle G, H\rangle$ is collapsing.*

Notice that the assumptions of the theorem that $NH = G$ or $N \cap H = \{1\}$ for every normal subgroup N of G imply, in particular, that H is a core-free subgroup of G. Thus, by Corollary 2.3, it is clear that Theorem 3.1 is the exact translation of Theorem 1.1 into a statement about G-algebras.

Before showing that the condition in Theorem 3.1 is sufficient for a pair $\langle G, H\rangle$ to be collapsing, we state and prove a proposition that provides some easy necessary conditions.

Proposition 3.2. *Let G be a finite group, and let H be a nontrivial proper subgroup of G. If the pair $\langle G, H\rangle$ is collapsing, then*
1. *H is a core-free subgroup of G;*
2. *H is contained in no proper normal subgroup of G.*

Proof. Recall from Section 2 that any G-operation $f(x, y_1, \ldots, y_n)$ is uniquely determined by the associated operation $\widetilde{f}(y_1, \ldots, y_n) = f(1, y_1, \ldots, y_n)$ where 1 is the unit element of the group G, and conversely, every operation $\widetilde{f}(y_1, \ldots, y_n)$ on G arises in this way from a G-operation $f(x, y_1, \ldots, y_n)$. The exact relationship between f and $\widetilde{f}$ is shown by equality (2.1) in Section 2.

Claim 3.3. *Let $f(x, y_1, \ldots, y_n)$ be an arbitrary G-operation and let $\widetilde{f}(y_1, \ldots, y_n)$ be the corresponding polynomial operation. Then*
- *f is idempotent if and only if $\widetilde{f}(1, \ldots, 1) = 1$;*
- *f is a projection if and only if $\widetilde{f}$ is either a projection or the constant function with value 1; and*
- *The algebra $(G; f)$ admits α_H as a congruence if and only if $\widetilde{f}$ preserves all equivalence relations α_{H^g} $(g \in G)$.*

The first two items are obvious from (2.1) and from the fact that if $(x, y_1, \ldots, y_n)$ runs over all $(n+1)$-tuples in G, then $(y_1x^{-1}, \ldots, y_nx^{-1})$ runs over all n-tuples in G.

To prove the last item assume first that $(G; f)$ admits α_H as a congruence. Since the permutations in R_G are automorphisms of $(G; f)$, therefore α_{H^g} is a congruence of $(G; f)$ for every conjugate H^g $(g \in G)$ of H. But $\widetilde{f}$ is a polynomial operation of $(G; f)$, hence it preserves all α_{H^g}.

Conversely, assume that $\widetilde{f}$ preserves all equivalence relations α_{H^g} $(g \in G)$. Let $(a, b_1, \ldots, b_n)$ and $(c, d_1, \ldots, d_n)$ be $(n+1)$-tuples from G which are α_H-related coordinatewise; that is, we have $aH = cH$ and $b_iH = d_iH$ for all i. We have to verify that $f(a, b_1, \dots, b_n)$ and $f(c, d_1, \dots, d_n)$ are also α_H-related, that is,

$$f(a, b_1, \dots, b_n)H = f(c, d_1, \dots, d_n)H.$$

The assumptions imply that $b_i a^{-1} aH = d_i c^{-1} cH = d_i c^{-1} aH$ for all i, or equivalently, $(b_i a^{-1}, d_i c^{-1}) \in \alpha_{H^{a^{-1}}}$ for all i. Since $\widetilde{f}$ preserves $\alpha_{H^{a^{-1}}}$, we conclude that

$$\widetilde{f}(b_1 a^{-1}, \dots, b_n a^{-1})aH = \widetilde{f}(d_1 c^{-1}, \dots, d_n c^{-1})aH = \widetilde{f}(d_1 c^{-1}, \dots, d_n c^{-1})cH.$$

(2.1) shows that this equality implies the previous displayed equality. This completes the proof of Claim 3.3.

We will prove the claims in Proposition 3.2 as follows: if condition (1) or (2) fails for G and H, we will exhibit an idempotent G-operation f which is distinct from the projections and admits α_H as a congruence. This will complete the proof because for such an f the algebra $(G; L_G, f)$ is a G-algebra which is not term equivalent to the unary algebra $(G; L_G)$, but which admits α_H as a congruence. We will construct f via $\widetilde{f}$. By Claim 3.3 the properties required for $\widetilde{f}$ are that (i) $\widetilde{f}(1, \dots, 1) = 1$, (ii) $\widetilde{f}$ is distinct from the projections and from the constant function with value 1, and (iii) $\widetilde{f}$ preserves all equivalence relations α_{H^g} $(g \in G)$.

Assume first that (1) fails, that is, G has a nontrivial normal subgroup M such that $M \subseteq H$. Let $\widetilde{f}$ be any function $G \to M$ such that $\widetilde{f}(1) = 1$ and $\widetilde{f}$ is not the constant function with value 1. Clearly, such a function exists and satisfies (i) and (ii). To see that (iii) also holds, observe that $M \subseteq H^g$ for all $g \in G$, therefore the range of $\widetilde{f}$ is contained in a single block of each equivalence relation α_{H^g}.

Now assume that (2) fails, that is, G has a proper normal subgroup N such that $H \subseteq N$. Clearly $H^g \subseteq N$ for all $g \in G$. Let us select an element from each coset of N in such a way that the element 1 is chosen from N. For each $z \in G$ let $\hat{z}$ denote the element selected from the coset zN. Define a binary operation $\widetilde{f}$ on G as follows:

$$\widetilde{f}(y, z) = \begin{cases} \hat{z} & \text{if } y \in N, \\ 1 & \text{if } y \notin N. \end{cases}$$

It is straightforward to check that $\widetilde{f}$ satisfies (i) and (ii). Since $\widetilde{f}(y, z) = \widetilde{f}(y', z')$ whenever $yN = y'N$ and $zN = z'N$ it follows also that $\widetilde{f}$ satisfies (iii). □

From now on, when we investigate whether a pair $\langle G, H \rangle$, where G is a finite group and H is a nontrivial subgroup of G, is collapsing we will always assume that conditions (1) and (2) from Proposition 3.2 are met. The following theorem is the main result of this section. This theorem gives a rather technical necessary and sufficient condition for $\langle G, H \rangle$ to be collapsing. However, the condition is strong enough to imply Theorem 3.1 (and hence Theorem 1.1). It will also be useful in the next section where we consider some examples to explore the gap between the necessary conditions in Proposition 3.2 and the sufficient condition provided by Theorem 3.1.

Theorem 3.4. *Let G be a finite group and let H be a nontrivial core-free subgroup of G such that H is contained in no proper normal subgroup of G. The following conditions are equivalent.*

(i) *$\langle G, H\rangle$ is a collapsing pair.*

(ii) *For every normal subgroup N of G such that $NH \neq G$ and $N \cap H \neq \{1\}$ the constant function with value 1 is the only function $\widetilde{f}\colon G \to N$ which has the following properties:*

  (a) *$\widetilde{f}(1) = 1$,*

  (b) *$\widetilde{f}$ preserves all equivalence relations α_{H^g} $(g \in G)$.*

Proof. Suppose that G and H satisfy the assumptions of the theorem. If condition (ii) fails, then for some nontrivial proper normal subgroup N of G there exists a nonconstant function $\widetilde{f}\colon G \to N$ with properties (a) and (b). Claim 3.3 shows that the binary G-operation f corresponding to $\widetilde{f}$ is idempotent, but is not a projection, and admits α_H as a congruence. Thus $(G; L_G, f)$ is a G-algebra which is not term equivalent to the unary algebra $(G; L_G)$, but it admits α_H as a congruence. This proves that (i) fails.

Now suppose that condition (ii) holds for G and H, and in order to prove (i) consider an arbitrary G-algebra $\mathbf{G}$ which admits α_H as a congruence. Our goal is to show that $\mathbf{G}$ is essentially unary, and hence term equivalent to the unary algebra $(G; L_G)$.

We know that every congruence of $\mathbf{G}$ is a congruence of the reduct $(G; L_G)$ of $\mathbf{G}$ as well, therefore it is of the form $\alpha_{\overline{H}}$ for some subgroup $\overline{H}$ of G. Select $\overline{H}$ so that $\alpha_{\overline{H}}$ is a maximal congruence of $\mathbf{G}$ above α_H. Thus the quotient algebra $\mathbf{G}/\alpha_{\overline{H}}$ is simple; furthermore, $\mathrm{Clo}_1\,(\mathbf{G}/\alpha_{\overline{H}}) = L_G[\overline{H}]$. Our assumptions on H ensure that $\overline{H}$ is not a normal subgroup of G, hence the permutation group $L_G[\overline{H}]$ is not regular. Therefore we are in a position to apply Lemma 10 from [3] (see [8] for a second proof):

Lemma 3.5. *Let Γ be a transitive permutation group acting on a finite set A, and assume that Γ is not regular. If $\mathbf{A}$ is a simple algebra such that $\mathrm{Clo}_1\,\mathbf{A} = \Gamma$, then $\mathbf{A}$ is essentially unary, hence $\mathbf{A}$ must be term equivalent to $(A; \Gamma)$.*

So we get from this lemma that the algebra $\mathbf{G}/\alpha_{\overline{H}}$ is essentially unary. Let N denote the intersection of all conjugates of $\overline{H}$; N is obviously a normal subgroup of G. Since both N and H are contained in $\overline{H}$, therefore $NH \neq G$. By Claim 2.2 the algebra $\mathbf{G}/\alpha_N$ is isomorphic to a subdirect power of $\mathbf{G}/\alpha_{\overline{H}}$. Hence $\mathbf{G}/\alpha_N$ is also essentially unary.

Next we want to show that every binary idempotent term operation $f(x, y)$ of $\mathbf{G}$ is a projection. Since $\mathbf{A}/\alpha_N$ is essentially unary, therefore f is a projection modulo α_N. We may assume without loss of generality that f projects onto the variable x modulo α_N. Hence $f(1, y)$ is α_N-related to 1 for all $y \in G$. In other words, this means that the operation $\widetilde{f}(y) = f(1, y)$ corresponding to f has range contained in N. Since

α_H is a congruence of $\mathbf{G}$, we know from Claim 3.3 that $\widetilde{f}$ preserves all equivalence relations α_{H^g} $(g \in G)$. The idempotence of f yields that $\widetilde{f}(1) = 1$.

If $N \cap H \neq \{1\}$, then the assumption (ii) combined with the properties of $\widetilde{f}$ established so far imply that $\widetilde{f}$ must be the constant function with value 1. The same conclusion is true also in the case when $N \cap H = \{1\}$. Indeed, this equality implies that $N \cap H^g = (N \cap H)^g = \{1\}$ for all $g \in G$. If $yH^g = zH^g$ for some elements $y, z, g \in G$, then $\widetilde{f}(y)H^g = \widetilde{f}(z)H^g$, as $\widetilde{f}$ preserves α_{H^g}. We have also $\widetilde{f}(y), \widetilde{f}(z) \in N$, as $\widetilde{f}$ maps into N. Thus $\widetilde{f}(y)^{-1}\widetilde{f}(z) \in N \cap H^g = \{1\}$, whence $\widetilde{f}(y) = \widetilde{f}(z)$. This means that $\widetilde{f}$ is constant on all blocks of each equivalence relation α_{H^g}. Consequently $\widetilde{f}$ is constant on all blocks of the smallest equivalence relation containing all α_{H^g} $(g \in G)$, which is the relation α_M where M is the least subgroup of G containing all H^g $(g \in G)$. This subgroup M is normal, therefore our assumption on H yields that $M = G$. Hence $\widetilde{f}$ is the constant function with value 1, as claimed.

Thus, in both cases $\widetilde{f}$ is the constant function with value 1. Using Claim 3.3 again we get that f is projection onto its variable x. Since f was an arbitrary binary idempotent term operation of $\mathbf{G}$, this shows that $\mathbf{G}$ has no binary idempotent term operations other than the projections.

To finish the proof of Theorem 3.4 it suffices to verify the claim below, since the assumptions of the theorem exclude the possibility that the group G is abelian.

Claim 3.6. *If $\mathbf{G}$ is a finite G-algebra and $\mathbf{G}$ has no binary idempotent term operations other than the projections, then $\mathbf{G}$ is term equivalent to one of the following algebras:*

(1) $(G; L_G)$,
(2) $(G; xy^{-1}z, L_G)$ *where G an elementary abelian 2-group,*
(3) $(G; m, L_G)$ *where $|G| = 2$ and m is a majority operation on G.*

Let $\mathbf{G}$ be a G-algebra. First we show that every unary polynomial operation of $\mathbf{G}$ is of the form $g\widetilde{f}$ for some $g \in G$ and some binary idempotent term operation f of $\mathbf{G}$. Let $p(x)$ be a unary polynomial operation of $\mathbf{G}$, and consider the polynomial operation $\overline{p}(x) = g^{-1}p(x)$ where $g = p(1)$. Clearly $\overline{p}(1) = 1$. Furthermore, $\overline{p}(x) = t(g_1, \dots, g_{n-1}, x)$ for some n, some n-ary term operation t of $\mathbf{G}$, and some elements $g_1, \dots, g_{n-1} \in G$. Hence for the binary term operation $f(x, y) = t(g_1x, \dots, g_{n-1}x, y)$ we have $\overline{p}(x) = f(1, x) = \widetilde{f}(x)$ and $\widetilde{f}(1) = \overline{p}(1) = 1$. Thus, $p(x) = g\overline{p}(x) = g\widetilde{f}(x)$ as claimed, and the equality $\widetilde{f}(1) = 1$ ensures by Claim 3.3 that f is idempotent. This completes the proof that any unary polynomial of a G-algebra $\mathbf{G}$ has the form $g\widetilde{f}$ for some $g \in G$ and some binary idempotent term operation f of $\mathbf{G}$.

Now assume that $\mathbf{G}$ is a finite G-algebra in which every binary idempotent term operation is a projection. The claim just proven shows that any unary polynomial of $\mathbf{G}$ is a permutation or a constant. So, using Pálfy's Theorem from [2], we get that either $\mathbf{G}$ has two elements, or $\mathbf{G}$ is essentially unary, or $\mathbf{G}$ is polynomially equivalent to a vector space. The conclusions of Claim 3.6 can be verified for the case $|G| = 2$ by consulting Post's classification [4] of all clones on a two-element set. Therefore from now on we assume that $|G| \geq 3$. Since $\mathbf{G}$ is a G-algebra, if it is essentially unary, then it is term equivalent to $(G; L_G)$.

Assume that $\mathbf{G}$ is polynomially equivalent to a vector space. In this case every idempotent term operation of the vector space is a term operation of $\mathbf{G}$ (see Exercise 2.8 and Proposition 2.6 in [7]). In particular, every operation $cx + (1 - c)y$ with c in the base field is a term operation of $\mathbf{G}$. Since $\mathbf{G}$ has no binary idempotent term operations other than the projections, we see that the base field must be the two-element field. Since $\mathbf{G}$ is a G-algebra, Proposition 2.9 in [7] shows also that $\mathbf{G}$ must be term equivalent to the algebra whose basic operations are the vector space operation $x - y + z$ and all translations. In this case G must be an elementary abelian 2-group and L_G must be the group of all translations of the vector space. Thus $\mathbf{G}$ is term equivalent to the algebra in (2).

This completes the proof of Claim 3.6 and hence the proof of Theorem 3.4. □

We note that Claim 3.6 could have been replaced by other arguments to finish the proof of (ii)⇒(i) in Theorem 3.4. Once it is established that $\mathbf{G}$ has no binary idempotent term operations other than the projections, one could apply Rosenberg's classification of minimal clones in [6] to prove that $\mathbf{G}$ is term equivalent to one of the algebras in 3.6 (1), (2), or (3), or one could cite Grabowski's theorem in [1] to directly conclude that $\langle G, H\rangle$ is collapsing.

Proof of Theorem 3.1. If the assumptions of Theorem 3.1 hold for a finite group G and its subgroup H, then the collection of normal subgroups N for which the condition in (ii) of Theorem 3.4 has to be verified is vacuous. Therefore (ii) automatically holds, and hence the pair $\langle G, H\rangle$ is collapsing. □

4. Examples

In this section we will look at some finite groups G which have subgroups H such that the sufficient condition in Theorem 3.1 fails in the simplest possible way: G has exactly one normal subgroup N with the properties $NH \neq G$ and $N \cap H \neq \{1\}$.

The smallest group that can be chosen for G is S_4, the symmetric group on four letters. Within S_4, H can be taken to be one of its four-element non-normal subgroups. Then, for any such H and for the four-element normal subgroup N of S_4 (the Klein group), we have $|NH| = 8 < |S_4|$ and $|N \cap H| = 2$. Since S_4 has no nontrivial proper normal subgroups other than N and the alternating group, it is clear that N is the only normal subgroup of G whose intersection with H is nontrivial, and whose join with H is a proper subgroup of G.

We will prove that these subgroups H in S_4 yield non-collapsing pairs, showing that if the hypotheses in Theorem 3.1 are not met, then the conclusion may not hold.

Proposition 4.1. *If H is a four-element subgroup of S_4 which is not normal, then the pair $\langle S_4, H\rangle$ is non-collapsing.*

Proof. Up to automorphism, there are two possibilities for H:

- H is generated by $(1\ 2\ 3\ 4)$, or
- $H = \{\mathrm{id},\ (1\ 3),\ (2\ 4),\ (1\ 3)(2\ 4)\}$.

In both of these cases NH is the usual permutation representation of the dihedral group D_4, and $N \cap H = \{\mathrm{id}, (1\ 3)(2\ 4)\}$. In view of Theorem 3.4 the proof will be done if we exhibit a nonconstant function $\widetilde{f}\colon S_4 \to N$ which satisfies conditions (a)–(b) in that theorem.

The function we construct will be constant on each of the six cosets of N. To visualize the construction we introduce an edge-colored graph $\mathcal{G}$. The vertices of $\mathcal{G}$ will be the cosets of N written as $a^i b^j N$ with $a = (1\ 2\ 3)$, $b = (1\ 3)$, $0 \le i \le 2$, $0 \le j \le 1$. The edges between vertices will be colored by conjugates of H. The edges are defined as follows: two cosets $yN, y'N$ are connected by an edge with color H^g $(g \in S_4)$ if

(4.1) there exist elements $s \in yN$ and $s' \in y'N$ such that $(s, s') \in \alpha_{H^g}$.

In other words, (4.1) requires that $\big((yN)^{-1}y'N\big)\cap H^g \neq \emptyset$. This condition is equivalent to $(Ny^{-1}y'N) \cap H^g \neq \emptyset$, which in turn is equivalent to $y^{-1}y' \in NH^gN = NH^g = H^gN$. Thus, two vertices $yN, y'N$ are connected by an edge with color H^g $(g \in S_4)$ if and only if the cosets $yN, y'N$ are contained in the same left coset of H^gN. The subgroup H has three distinct conjugates in S_4, namely H, H^a, and H^{a^2}, therefore $\mathcal{G}$ is the graph depicted in Figure 1, where thick, normal, and dotted edges correspond to edges with color H, H^a, and H^{a^2}, respectively. Note that $\mathcal{G}$ has loops of each color at all its vertices which are not shown in Figure 1.

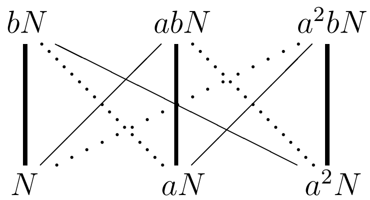

FIGURE 1: The graph $\mathcal{G}$ (loops are not indicated)

We can think of the partitions α_{H^g} $(g \in S_4)$ of the group as a 'geometry' where the points are the elements of S_4 and the lines are the blocks of these partitions, i.e., the lines are the left cosets of H^g $(g \in S_4)$. The graph $\mathcal{G}$ describes this geometry modulo N. The graph $\mathcal{G}_N$ in Figure 2 shows this geometry restricted to N: the vertices are the elements id, $u = (1\ 2)(3\ 4)$, $v = (1\ 3)(2\ 4)$, $w = (1\ 4)(2\ 3)$ of N, and there is an edge with color H^g ($g \in S_4$) between two vertices exactly when they are α_{H^g}-related. Again, there are loops of each color at all vertices, which are not shown in Figure 2.

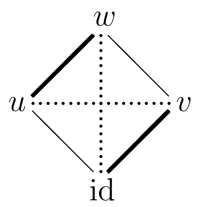

FIGURE 2: The graph $\mathcal{G}_N$ (loops are not indicated)

It is clear from the definitions of the graphs $\mathcal{G}$ and $\mathcal{G}_N$ that a function $\widetilde{f}\colon S_4 \to N$ which is constant on each coset of N has property (b) in Theorem 3.4 if and only if it is a color-preserving function from $\mathcal{G}$ to $\mathcal{G}_N$. Such a function can be viewed as a color-preserving labelling of the vertices of $\mathcal{G}$ with the vertices of $\mathcal{G}_N$. Thus, the labelling shown in Figure 3 provides a nonconstant function $\widetilde{f}$ which completes the proof.

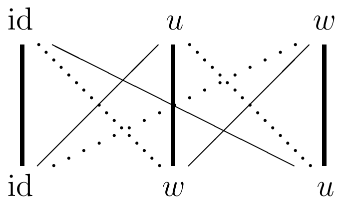

FIGURE 3: Color-preserving labelling of $\mathcal{G}$

□

Next we will look at a class of groups constructed from finite fields. Let m be a prime number of the form $m = (p^q - 1)/(p-1)$ where p is prime. It is straightforward to verify that q must be prime and $\gcd(p-1, q) = \gcd(p^j - 1, m) = 1$ for $0 < j < q$. (There are many choices of p and q which make m prime. For example, Mersenne primes m arise for $p = 2$. A computer search shows that m is prime for many other values of p and q.)

We define a group G whose elements are permutations of the Galois field $\mathrm{GF}(p^q)$:

$$G = \{cx^{p^j} + d : c, d \in \mathrm{GF}(p^q),\ c^m = 1,\ 0 \le j < q\}. \tag{4.2}$$

The following subgroups of G will be of importance:

$$\begin{aligned} H &= \{x^{p^j} + n : 0 \le j < q,\ n \in \mathrm{GF}(p)\}, \\ N &= \{x + d : d \in \mathrm{GF}(p^q)\}. \end{aligned} \tag{4.3}$$

Using the fact that m is prime one can verify that G has only two nontrivial proper normal subgroups: N and

$$A = \{cx + d : c, d \in \mathrm{GF}(p^q),\ c^m = 1\}.$$

Therefore N is the only normal subgroup of G whose intersection with H is nontrivial, and whose join with H is a proper subgroup of G.

Notice that $m = 3$ is the least one among the primes we are considering, and it arises for $p = q = 2$. In this case the group G in (4.2) is isomorphic to S_4, and the subgroup H in (4.3) corresponds to the noncyclic four-element subgroup of S_4 that we considered earlier, while N corresponds to the Klein group and A corresponds to A_4. Thus, S_4 is the smallest member of the family we are considering. However, we will see that S_4 is an exceptional member of the family: all groups with $m > 3$ yield collapsing pairs.

Proposition 4.2. *If $m > 3$ is a prime of the form $m = (p^q - 1)/(p - 1)$ where p is prime, then for the groups G, H defined in* (4.2) *and* (4.3) *the pair $\langle G, H\rangle$ is collapsing.*

Proof. By Theorem 3.4 we have to prove that the constant function with value 1 is the only function $\widetilde{f}\colon G \to N$ which satisfies conditions (a)–(b) in that theorem. The crucial condition is (b), so we will analyze what it means for a function $\widetilde{f}\colon G \to N$ to satisfy this condition. First we will focus on the behavior of $\widetilde{f}$ when $\widetilde{f}$ is restricted to the subgroup

$$G_0 = \{cx^{p^j} : c \in \mathrm{GF}(p^q),\ c^m = 1,\ 0 \le j < q\},$$

which is the stabilizer of the element $0 \in \mathrm{GF}(p^q)$ in G. To this end we will study how the geometry of left cosets of conjugates of H restricts to G_0 and to N.

To describe these geometries in detail we need some notation. Let

$$H_0 = G_0 \cap H = \{x^{p^j} : 0 \le j < q\} \qquad \text{and} \qquad A_0 = \{cx : c \in \mathrm{GF}(p^q),\ c^m = 1\},$$

that is, H_0 and A_0 are the stabilizers of 0 in H and A, respectively. Clearly, A_0 is a normal subgroup of G_0 and $G_0 = A_0H_0$. Since $A_0 \cap H_0 = \{1\}$, G_0 is a semidirect product of A_0 and H_0.

Let a denote a generating element of the cyclic group A_0, and let $b = x^p$, which is a generating element of the cyclic group H_0. Further, let 1 denote the identity permutation. Then $\{a, b\}$ is a generating set of G_0, and

$$\text{(4.4)} \qquad a^m = 1, \quad b^q = 1, \quad b^j a^i = a^{ip^j} b^j \quad \text{for all} \quad 0 \le i < m,\ 0 \le j < q.$$

Every element of G_0 can be written uniquely in the form $a^i b^j$ for some $0 \le i < m$, $0 \le j < q$.

Claim 4.3. *$H \cap H^{a^k} = \{1\}$ if $0 < k < m$. Hence the conjugates H^{a^k} $(0 \le k < m)$ of H pairwise intersect trivially.*

The second statement is an easy consequence of the first one. To prove the first statement let $0 < k < m$. The permutation a^k is of the form cx for some element $c \in \mathrm{GF}(p^q)$ with $c^m = 1$, $c \neq 1$. Since every permutation in H has the form $x^{p^j} + n$ where $0 \leq j < q$ and $n \in \mathrm{GF}(p)$, it follows that every permutation in H^{a^k} has the form

$$c^{-1}((cx)^{p^j} + n) = c^{p^j-1}x^{p^j} + c^{-1}n$$

for some $0 \leq j < q$ and $n \in \mathrm{GF}(p)$. Thus every member of $H \cap H^{a^k}$ has the form $c^{p^j-1}x^{p^j} + c^{-1}n$ with $c^{p^j-1} = 1$ and $c^{-1}n \in \mathrm{GF}(p)$. But $c^{p^j-1} \neq 1$ unless $j = 0$, because otherwise we contradict $c^m = 1$, $c \neq 1$, or the fact that $\gcd(p^j - 1, m) = 1$ for $0 < j < q$. Thus $j = 0$. This also shows that $c \notin \mathrm{GF}(p)$, since otherwise c would satisfy $c^{p-1} = 1$. Thus, since $n, c^{-1}n \in \mathrm{GF}(p)$, we conclude that $n = 0$. This shows that every permutation in $H \cap H^{a^k}$ $(0 < k < m)$ has the form

$$c^{p^j-1}x^{p^j} + c^{-1}n = c^{p^0-1}x^{p^0} + 0 = x,$$

which is the identity permutation. This completes the proof of the claim.

Now we will look at how the left cosets of H^{a^k} $(0 \leq k < m)$ restrict to G_0. We will think of this structure as a geometry $\mathcal{G}_0$ on G_0. The points of the geometry are the elements a^ib^j $(0 \leq i < m,\ 0 \leq j < q)$ arranged in a $q \times m$ rectangle so that the i-th column is the left coset a^iH_0 of H_0 and the j-th row is the right coset A_0b^j of A_0. The lines of the geometry are the restrictions of the left cosets of H^{a^k} $(0 \leq k < m)$ to G_0, and each line is colored by the associated group H^{a^k}. In other words, the lines with color H^{a^k} are the sets of the form

$$G_0 \cap yH^{a^k} = y(G_0 \cap H^{a^k}) = y(G_0 \cap H)^{a^k} = yH_0^{a^k} \quad \text{with} \quad y \in G_0. \tag{4.5}$$

Some basic properties of the geometry $\mathcal{G}_0$ are summarized in Claim 4.4 below.

Claim 4.4. *The geometry $\mathcal{G}_0$ has the following properties:*

(G1) *Every point is incident to a line of each color.*
(G2) *Two distinct lines of the same color have no points in common.*
(G3) *Two lines of different colors have at most one point in common.*
(G4) *Every line contains exactly q points, one point from each row.*
(G5) *Any two points which are not in the same row lie on exactly one line.*

(G1) and (G2) are obvious, since the cosets of any $H_0^{a^k}$ partition G_0. To prove (G3) let us consider two lines of different colors which have a point $y \in G_0$ in common. Using the description (4.5) of the lines we see that the two lines are of the form $L = G_0 \cap yH^{a^k}$ and $L' = G_0 \cap yH^{a^{k'}}$ for some $0 \leq k < k' < m$. Thus

$$L \cap L' = G_0 \cap (yH^{a^k} \cap yH^{a^{k'}}) = G_0 \cap y(H^{a^k} \cap H^{a^{k'}}) = G_0 \cap \{y\} = \{y\}$$

by Claim 4.3.

(G4) follows from the facts that the lines are the left cosets of the subgroups $H_0^{a^k}$ in G_0 (by (4.5)), the rows are the right cosets of A_0 in G_0, G_0 is a semidirect product of the normal subgroup A_0 by H_0, and $|H_0| = q$.

To show (G5) pick points P and Q from different rows. Suppose P is from row A_0b^s and Q is from row A_0b^t and $s \neq t$. Then $P^{-1}Q$ belongs to $b^{-s}A_0b^t = A_0b^{t-s}$. Since $s \neq t$, $P^{-1}Q$ has the form $P^{-1}Q = a^ib^j$ for some $0 \leq i < m$ and $0 < j < q$. By the description (4.5) of the lines, P and Q belong to a line with color H^{a^k} exactly when $P^{-1}Q \in H_0^{a^k}$. Using the fact that

$$H_0^{a^k} = \{a^{k(p^j-1)}b^j : 0 \leq j < q\}, \qquad 0 \leq k < m,$$

which follows from (4.4), we see that $P^{-1}Q = a^ib^j \in H_0^{a^k}$ holds if and only if

$$k(p^j - 1) \equiv i \pmod{m}. \tag{4.6}$$

Since $\gcd(p^j - 1, m) = 1$ holds for $0 < j < q$, therefore the congruence (4.6) has a unique solution for k. Thus P and Q lie on a unique line. This completes the proof of Claim 4.4.

We will also need the geometry $\mathcal{G}_N$ that arises by restricting the geometry of left cosets of H^{a^k} $(0 \leq k < m)$ to N. The points of this geometry are the elements of N. The lines of the geometry are the restrictions of the left cosets of H^{a^k} $(0 \leq k < m)$ to N, and each line is colored by the associated group H^{a^k}. In other words, the lines with color H^{a^k} are the sets of the form

$$N \cap uH^{a^k} = u(N \cap H^{a^k}) = u(N \cap H)^{a^k} \quad \text{with} \quad u \in N. \tag{4.7}$$

Claim 4.5. *$\mathcal{G}_N$ is the q-dimensional affine geometry over the field* GF(p). *In particular, the following conditions hold:*

(N1) *Every point is incident to a line of each color.*
(N2) *Two distinct lines of the same color have no points in common.*
(N3) *Two lines of different colors have at most one point in common.*
(N4) *Every line contains exactly p points.*
(N5) *Any two distinct points lie on exactly one line.*

Clearly, N is an elementary abelian p-group of order p^q, which can be considered to be a q-dimensional vector space over GF(p). Conjugation by the elements a^k $(0 \leq k < m)$ are linear transformations of N. Furthermore, in this space $N \cap H$ is a one-dimensional subspace. The conjugates $(N \cap H)^{a^k} = N \cap H^{a^k}$ $(0 \leq k < m)$ pairwise intersect trivially by Claim 4.3, so these conjugates of $N \cap H$ are pairwise distinct one-dimensional subspaces of N. Since there are m conjugates of $N \cap H$ and m one-dimensional subspaces of N, it follows that the conjugates $(N \cap H)^{a^k}$ $(0 \leq k < m)$ of $N \cap H$ are exactly the one-dimensional subspaces of N. This implies that $\mathcal{G}_N$ is the q-dimensional affine geometry over GF(p). All items (N1)–(N5) follow from this.

Now we will consider functions $\lambda\colon G_0 \to N$ which preserve the equivalence relations $\alpha_{H^{a^k}}$ for all k ($0 \le k < m$). Geometrically, these are exactly the color-preserving functions which map the points of the geometry $\mathcal{G}_0$ into the points of the geometry $\mathcal{G}_N$. We will refer to these functions as color-preserving functions $\mathcal{G}_0 \to \mathcal{G}_N$.

Claim 4.6. *Every color-preserving function $\mathcal{G}_0 \to \mathcal{G}_N$ is constant.*

Our argument will be based exclusively on the properties of the geometries $\mathcal{G}_0$ and $\mathcal{G}_N$ listed in Claims 4.4 and 4.5. From the definition of the coloring it is essential that we use the same set of colors in the two geometries, but it is irrelevant that the colors come from groups H^{a^k}; therefore we will use colors like 'red', 'blue', etc., and in Figures 4 and 5 different colors will be represented by different kinds of lines (continuous, dotted). Notice that by properties (G2)–(G3) and (N2)–(N3), every line in each geometry $\mathcal{G}_0$ and $\mathcal{G}_N$ has a uniquely determined color, so there is no ambiguity in talking about *the* color of a line.

Let $\lambda\colon \mathcal{G}_0 \to \mathcal{G}_N$ be a color-preserving function. If X is a point in $\mathcal{G}_0$, then we will refer to $\lambda(X)$ as the label of X.

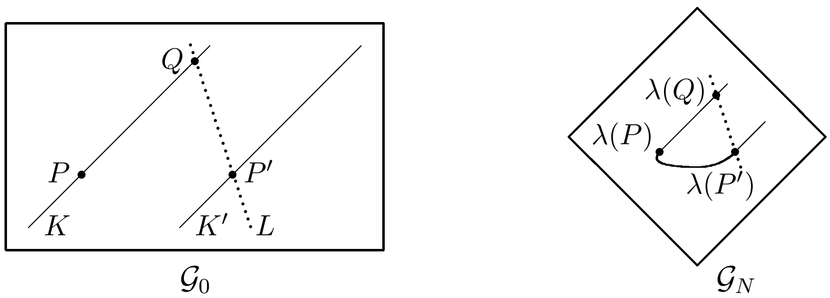

Figure 4

Suppose that there are two distinct points $P, P' \in \mathcal{G}_0$ in the same row such that $\lambda(P) \ne \lambda(P')$. By (N5), there is a line through $\lambda(P)$ and $\lambda(P')$ in $\mathcal{G}_N$. Call the color of this line red. Now consider the red lines K and K' incident to P and P', respectively, in $\mathcal{G}_0$. Such lines exist by (G1), and are distinct, and hence disjoint by (G4) and (G2). Let Q be any point of K distinct from P. By (G4) Q is in a different row than P and P', therefore by (G5) there is a unique line L containing Q and P', and by (G2) its color is not red. Call the color of L blue. Since λ is color-preserving, therefore the labels $\lambda(P'), \lambda(P), \lambda(Q)$ are on the same red line in $\mathcal{G}_N$, and $\lambda(P'), \lambda(Q)$ are also on the same blue line in $\mathcal{G}_N$. (See Figure 4.) By (N3), in $\mathcal{G}_N$ two lines with different colors can have at most one point in common, so we conclude that $\lambda(Q) = \lambda(P')$. Since $Q \ne P$ can be chosen arbitrarily on K, this proves that every point of the line K, except P, has the same label as P'. Switching the roles of P and P' we get that every point of the line K', except P', has the same label as P.

Now let us fix a point Q on K which is distinct from P, and using (G4) find the point Q' on K' which is in the same row as Q. From what we proved in the preceding paragraph it follows that $\lambda(Q) = \lambda(P') \neq \lambda(P) = \lambda(Q')$. Therefore we can repeat the argument of the preceding paragraph for Q, Q' in place of P, P' to conclude that every point of the line K, except Q, has the same label as Q', and every point of the line K', except Q', has the same label as Q. This shows that every point of $K \setminus \{P\}$ has label $\lambda(P')$, and also every point of $K \setminus \{Q\}$ has label $\lambda(Q') = \lambda(P) \neq \lambda(P')$. Since the line K has q points, this is impossible unless $q = 2$. But $q = 2$ implies that $m = p+1$. Since m and p are prime, this forces $p = 2$ and $m = 3$. However, the case $m = 3$ is excluded by our assumption. So this contradiction shows that λ is constant on every row of points in $\mathcal{G}_0$.

Now let P, P' and R be arbitrary points in $\mathcal{G}_0$ such that P, P' are in the same row, and R is in a different row. By (G5) there is a unique line L incident to P and R, and a unique line L' incident to P' and R. By (G2) L and L' have different colors, so let L be green and L' be yellow, say. (See Figure 5.) We know from the previous paragraph that $\lambda(P) = \lambda(P')$. Since the function λ is color-preserving, it follows also that $\lambda(P), \lambda(R)$ are on the same green line in $\mathcal{G}_N$, and $\lambda(P'), \lambda(R)$ are on the same yellow line in $\mathcal{G}_N$. But then the two points $\lambda(P) = \lambda(P')$ and $\lambda(R)$ are incident to two lines of different colors. So it follows from (N3) that $\lambda(P) = \lambda(R)$. This shows that the labels on different rows are the same. Hence λ is constant, concluding the proof of Claim 4.6.

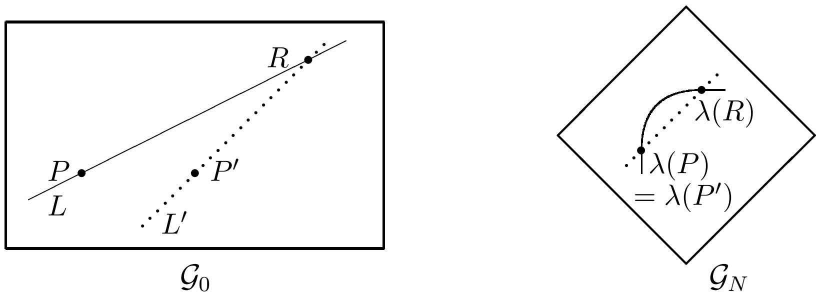

FIGURE 5

Now we are in a position to complete the proof of Proposition 4.2. Let $\widetilde{f}\colon G \to N$ be a function satisfying conditions (a)–(b) in Theorem 3.4. We have to prove that $\widetilde{f}$ is a constant function. By (a) it will follow then that it is the constant function with value 1.

The collection of equivalence relations α_{H^g} $(g \in G)$ is invariant under the permutations in L_G, and also under conjugation by elements of G. Therefore, together with the function $\widetilde{f}(x)$, every function $\widetilde{f}(g^{-1}x)$ and $\widetilde{f}(hxh^{-1})$ $(g, h \in G)$ will also satisfy condition (b). By combining the two transformations (left translation and conjugation) we see that every function of the form $\widetilde{f}(g^{-1}(hxh^{-1}))$ with $(g, h \in G)$ will satisfy condition (b). In particular, all these functions, when restricted to G_0, will yield a color-preserving function $\mathcal{G}_0 \to \mathcal{G}_N$. Hence, by Claim 4.6, all of these functions are constant. Consequently $\widetilde{f}$ is constant on every left coset of every conjugate of G_0, or equivalently, $\widetilde{f}$ is constant on all blocks of each equivalence relation $\alpha_{G_0^h}$ $(h \in G)$. This implies that $\widetilde{f}$ is constant on all blocks of the smallest equivalence relation containing all $\alpha_{G_0^h}$ $(h \in G)$, which is the relation α_M where M is the least subgroup of G containing all G_0^h $(h \in G)$. This subgroup M must be normal, and must contain G_0, therefore $M = G$. Hence $\widetilde{f}$ is a constant function, as claimed. $\square$

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