CALCULUS 3

November 12, 2008

3rd TEST

YOUR NAME:

- ◯ **001** B. KATZ-MOSES .... (8AM)
- ◯ **002** J. SANDERS .... (9AM)
- ◯ **003** J. NEWHALL .... (10AM)
- ◯ **004** A. SPINA .... (11AM)
- ◯ **005** E. ANGEL .... (12PM)
- ◯ **006** A. SPINA .... (1PM)
- ◯ **007** A. SPINA .... (3PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	10 pts	
3	10 pts	
4	15 pts	
5	10 pts	
6	10 pts	
7	15 pts	
8	15 pts	
TOTAL	100 pts	

1. **[15 pts]** Let $f(x,y) = x^2+4xy+y^2$, and let R be the region enclosed by a circle of radius 2, $x^2+y^2=4$. Find the absolute extrema of f over R.

SOLUTION:

We first look now for critical points inside the given domain,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x+4y = 0 \\ \dfrac{\partial f}{\partial y} = 4x+2y = 0 \end{cases} \qquad \Rightarrow \qquad (x,y)_c = (0,0)\,.$$

We also need to find the critical points on the boundary $x^2+y^2=4$. This boundary can be parametrized by

$$x_B = 2\cos\theta\,, \qquad y_B = 2\sin\theta\,,$$

and on this boundary

$$\begin{aligned} f(x_B,y_B) &= x_B^2 + 4x_By_B + y_B^2 \\ &= 4\cos^2\theta + 16\cos\theta\sin\theta + 4\sin^2\theta \\ &= 4 + 16\cos\theta\sin\theta\,. \end{aligned}$$

The critical points of this one-variable function of the parameter θ are given by the solutions of

$$\frac{\mathrm{d}}{\mathrm{d}\theta} f(x_B,y_B) = 0 \qquad \Rightarrow \qquad -\sin^2\theta + \cos^2\theta = 0 \qquad \Rightarrow \qquad \cos\theta = \pm\sin\theta\,,$$

that is,

$$\begin{aligned} \theta = +\frac{\pi}{4} &\quad \rightarrow \quad (x,y)_B = (+\sqrt{2},+\sqrt{2}) \\ \theta = -\frac{\pi}{4} &\quad \rightarrow \quad (x,y)_B = (+\sqrt{2},-\sqrt{2}) \\ \theta = +\frac{3\pi}{4} &\quad \rightarrow \quad (x,y)_B = (-\sqrt{2},+\sqrt{2}) \\ \theta = -\frac{3\pi}{4} &\quad \rightarrow \quad (x,y)_B = (-\sqrt{2},-\sqrt{2}) \end{aligned}$$

The values of the function at the critical points we found are

$$\begin{aligned} f(0,0) &= 0\,, \\ f(+\sqrt{2},+\sqrt{2}) &= 12\,, \\ f(+\sqrt{2},-\sqrt{2}) &= -4\,, \\ f(-\sqrt{2},+\sqrt{2}) &= -4\,, \\ f(-\sqrt{2},-\sqrt{2}) &= 12\,. \end{aligned}$$

Therefore

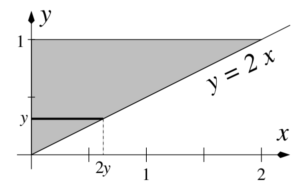
$$\boxed{f_{\max} = 12 \quad \text{at} \quad (x,y) = (\pm\sqrt{2},\pm\sqrt{2})\,, \qquad f_{\min} = -4 \quad \text{at} \quad (x,y) = (\pm\sqrt{2},\mp\sqrt{2})}$$

2. **[10 pts]** By changing the order of integration, evaluate

$$\int_{x=0}^{2}\int_{y=x/2}^{1} e^{y^2}\,\mathrm{d}y\,\mathrm{d}x\,.$$

SOLUTION:

We can read the new limits from the figure on the right. Changing the order of integration we have



$$\begin{aligned}\int_{x=0}^{2}\int_{y=x/2}^{1} e^{y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_{y=0}^{1}\int_{x=0}^{2y} e^{y^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{y=0}^{1}\left[x e^{y^2}\right]_{x=0}^{2y}\,\mathrm{d}y\\ &= \int_{y=0}^{1} 2y e^{y^2}\,\mathrm{d}y\\ &= \left[e^{y^2}\right]_{y=0}^{1}\\ &= e-1\,.\end{aligned}$$

3. **[10 pts]** Evaluate

$$\int_{y=-2}^{+2}\int_{x=-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \frac{1}{(1+x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y$$

SOLUTION:

$$\begin{aligned}\int_{y=-2}^{+2}\int_{x=-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \frac{1}{(1+x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\frac{1}{(1+r^2)^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \int_{\theta=0}^{2\pi}\left[-\frac{1}{2(1+r^2)}\right]_{r=0}^{2}\mathrm{d}\theta\\ &= \int_{\theta=0}^{2\pi}\frac{2}{5}\,\mathrm{d}\theta\\ &= \frac{4}{5}\pi\,.\end{aligned}$$

Therefore,

$$\boxed{\int_{y=-2}^{+2}\int_{x=-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} \frac{1}{(1+x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{4}{5}\pi}$$

4. **[15 pts]** Find the surface area of the portion of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2+y^2=1$, $x^2+y^2=4$.

SOLUTION:

The surface can be described in rectangulr coordinates by an expression of the form $z = f(x,y)$, so the simplest approach will be to start with

$$S = \iint_{\mathcal{R}} \left\|\frac{\partial \mathbf{r}}{\partial x}\times\frac{\partial \mathbf{r}}{\partial y}\right\|\,\mathrm{d}A_{xy} = \iint_{\mathcal{R}}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\;\mathrm{d}A_{xy}$$

with

$$\mathcal{R} = \left\{(x,y)\ :\ 1\le x^2+y^2\le 4\right\}$$

and, if needed, rewrite the resulting integral in a more appropriate system of coordinates in order to evaluate it.

The equation of the surface in rectangular coordinates is

$$z = f(x, y) = y^2 - x^2$$

and the surface area

$$\begin{aligned} S &= \iint_{\mathcal{R}} \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\ \mathrm{d}A_{xy} \\ &= \iint_{\mathcal{R}} \sqrt{(-2x)^2 + (2y)^2 + 1}\ \mathrm{d}A_{xy} \\ &= \iint_{\mathcal{R}} \sqrt{4x^2 + 4y^2 + 1}\ \mathrm{d}A_{xy} \end{aligned}$$

At this point is obvious that the evaluation is better done in polar coordinates. The integration limits in polar coordinates will be

$$\begin{aligned} 1 &\le r < 2\,, \\ 0 &\le \theta < 2\pi\,, \end{aligned}$$

and the integral takes the form

$$\begin{aligned} S &= \iint_{\mathcal{R}} \sqrt{4x^2 + 4y^2 + 1}\ \mathrm{d}A_{xy} \\ &= \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \sqrt{4r^2 + 1}\ r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_{\theta=0}^{2\pi} \left[\frac{1}{12}\left(4r^2 + 1\right)^{3/2}\right]_{r=1}^{2} \mathrm{d}\theta \\ &= \int_{\theta=0}^{2\pi} \frac{1}{12}\left(17^{3/2} - 5^{3/2}\right) \mathrm{d}\theta \\ &= \frac{1}{12}\left(17^{3/2} - 5^{3/2}\right) 2\pi\,. \end{aligned}$$

Therefore,

$$\boxed{S = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

5. **[10 pts]** Rewrite the integral

$$\int_{x=-1}^{1} \int_{y=x^2}^{1} \int_{z=0}^{1-y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

as an iterated integral in the order $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

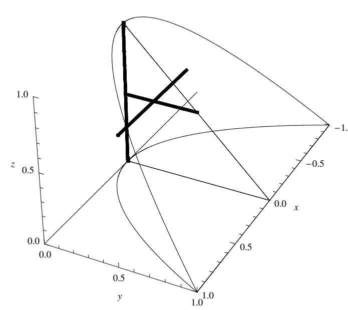
SOLUTION:

The safest way is to sketch the region of integration and then, as shown in the figure on the right,

(a) Take a point in its interior, and displace it along the x-direction until it hits the boundaries (in this case the parabolic cylinder $y = x^2$). This way we obtain $x_{\min}(y, z)$ and $x_{\max}(y, z)$.

(b) Colapse the x-axis. This yields the triangle shown in the yz-pane. Take a point there and move it along the y-direction until it hits the boundaries. This way we obtain $y_{\min}(z)$ and $y_{\max}(z)$.

(c) Colapse the y-axis of the triangle. This way we obtain the portion of the z-axis going between $z_{\min}$ and $z_{\max}$.

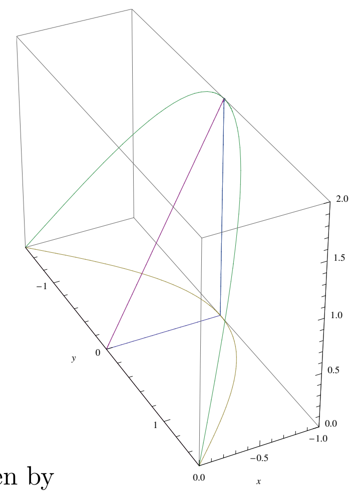


In this way we obtain

$$\boxed{\int_{x=-1}^{1}\int_{y=x^2}^{1}\int_{z=0}^{1-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=-\sqrt{y}}^{+\sqrt{y}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$$

6. **[10 pts]** You need to find the volume of the wedge-shaped region enclosed on the side by the cylinder $x = -\cos y$, $-\pi/2 \le y \le \pi/2$, on top by the plane $z = -2x$, and below by the xy-plane. Write the volume as a triple integral, clearly indicating the limits of integration and the order of integration. **Do not evaluate the resulting integral**.

   NOTE: Only one order of integration will give you the simplest possible integral. It is your task to find it!



SOLUTION:

To compute the volume we need to evaluate

$$V = \iiint_{\mathcal{V}} \mathrm{d}V$$

where the boundaries of the region $\mathcal{V}$ in cylindrical coordinates are given by

$$\mathcal{V} : \begin{cases} 0 \le z \le -2x\,, \\ -\cos y \le x \le 0\,, \\ -\pi/2 \le y \le \pi/2\,. \end{cases}$$

Substituting into the expression above we have

$$\boxed{V = \int_{y=-\pi/2}^{\pi/2}\int_{x=0}^{-\cos y}\int_{z=0}^{-2x} \mathrm{d}z\;\mathrm{d}x\;\mathrm{d}y}$$

7. **[15 pts]** You need to find the volume of the solid that is bounded on top by the sphere $\rho = a$, and below by the cone $\phi = \alpha$. Write the volume as a triple integral on an appropriate coordinate system other than the rectangular (where the integration becomes very messy), clearly indicating the limits of integration and the order of integration. **Evaluate the integral**.

SOLUTION:

The limits are already given in spherical coordinates, which are the most appropriate for the task The boundaries of the region $\mathcal{V}$ in spherical coordinates are given by

$$\boxed{\mathcal{V} : \begin{cases} 0 \le \rho \le a\,, \\ 0 \le \theta \le 2\pi\,, \\ 0 \le \phi \le \alpha\,. \end{cases}}$$

Therefore

$$\boxed{V = \int_{\phi=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi}$$

Evaluating the integral, we get

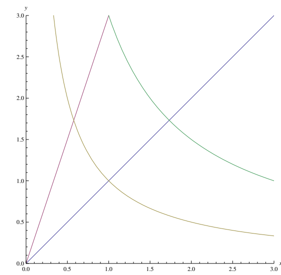
$$\begin{aligned} V &= \int_{\phi=0}^{\alpha}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_{\phi=0}^{\alpha}\int_{\theta=0}^{2\pi} \left(\frac{a^3}{3}\right) \sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{a^3}{3}\int_{\phi=0}^{\alpha} (2\pi)\sin\phi\,\mathrm{d}\phi \\ &= \frac{a^3}{3}\,2\pi\,(1-\cos\alpha)\,. \end{aligned}$$

Therefore,

$$\boxed{V = \frac{2\pi a^3}{3}\,(1-\cos\alpha)}$$

8. **[15 pts]** Evaluate the integral

$$\iint_{\mathcal{R}} xy\,\mathrm{d}A_{xy}$$



over the region $\mathcal{R}$ in the first quadrant enclosed by the curves $y=x$, $y=3x$, $xy=1$, and $xy=3$.
HINT: Use an appropriate transformation of coordinates.

SOLUTION:

The integrand is not complicated. On the other hand the domain of integration in the in the x and y variables has some complications. Let us look for a transformation whose grid-lines will match the given region of integration

$$\begin{cases} u &= y/x \\ v &= xy \end{cases} \qquad \Rightarrow \qquad \begin{cases} x &= \sqrt{v/u} \\ y &= \sqrt{u\,v} \end{cases}$$

will simplify the domain of integration which, in the new variables, is given by

$$\mathcal{S}\;:\;\begin{cases} u \in [1,3]\,, \\ v \in [1,3]\,. \end{cases}$$

The Jacobian of the transformation is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}u^{-3/2}v^{1/2} & \frac{1}{2}u^{-1/2}v^{-1/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = -\frac{1}{2u}\,,$$

and the integrand in the new variables is given by

$$x\,y \qquad \longrightarrow \qquad x(u,v)\,y(y,v) = \sqrt{\frac{v}{u}}\sqrt{uv} = v\,.$$

Therefore, the integral in the new variables will be

$$\begin{aligned}
\iint_{\mathcal{R}} xy\,\mathrm{d}A_{\mathcal{R}} &= \iint_{\mathcal{S}} x(u,v)y(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}A_{\mathcal{S}} \\
&= \iint_{\mathcal{S}} v\,\frac{1}{u}\,\mathrm{d}A_{\mathcal{S}} \\
&= \int_{u=1}^{3}\int_{v=1}^{3}\frac{v}{u}\,\mathrm{d}v\,\mathrm{d}u \\
&= \int_{u=1}^{3}\frac{2}{u}\,\mathrm{d}u \\
&= 2\,[\ln u]_{u=1}^{3} \\
&= \mathrm{d}A_{xy} = 2\ln 3\,.
\end{aligned}$$

Therefore,

$$\boxed{\iint_{\mathcal{R}} xy\,\mathrm{d}A_{xy} = 2\ln 3}$$