MATH 2400: CALCULUS 3

April 18, 2007

MIDTERM 3

I have neither given nor received aid on this exam.

Name:____

- ◯ **001** E. Kim .... (9am)
- ◯ **002** E. Angel .... (10am)
- ◯ **003** I. Mishev .... (11am)
- ◯ **004** M. Daniel .... (12am)
- ◯ **005** A. Gorokhovsky ...... (1pm)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	14 pts	
2	12 pts	
3	14 pts	
4	12 pts	
5	12 pts	
6	12 pts	
7	24 pts	
TOTAL	100 pts	

1. (14 pt) Compute $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$

Convert to polar:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 r\,dr\,d\theta = \frac{\pi}{8}$$

2. (12 pt) Find the volume of the solid that lies inside the sphere $\rho = 3$ and above the cone $r = z$ ($\phi = \frac{\pi}{4}$).

Use spherical:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 9 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin\phi\,d\phi\,d\theta = 18\pi - 9\pi\sqrt{2}$$

3. (14 pt) Find the volume of the solid bounded by

$$z = 6, \quad z = 2y, \quad y = x^2, \quad y = 2 - x^2 .$$

$$\begin{aligned} V &= \int_{-1}^{1} \int_{x^2}^{2-x^2} \int_{2y}^{6} dz\, dy\, dx \\ &= \int_{-1}^{1} [6(2-x^2) - (2-x^2)^2] - [6x^2 - x^4]\, dx \\ &= \int_{-1}^{1} 8 - 8x^2\, dx \\ &= \frac{32}{3} \end{aligned}$$

4. (12 pt) Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

We are concerned with the portion of the surface ABOVE the xy-plane, so the cone can be represented by $z = \sqrt{4x^2 + 4y^2}$. Then,

$$\begin{aligned} SA &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_R \sqrt{\left(\frac{4x}{\sqrt{4x^2+4y^2}}\right)^2 + \left(\frac{4y}{\sqrt{4x^2+4y^2}}\right)^2 + 1}\, dA \\ &= \iint_R \sqrt{\left(\frac{16x^2}{4x^2+4y^2}\right) + \left(\frac{16y^2}{4x^2+4y^2}\right) + 1}\, dA \\ &= \iint_R \sqrt{5}\, dA \\ &= \int_0^1 \int_{x^2}^{x} \sqrt{5}\, dy\, dx \\ &= \frac{\sqrt{5}}{6} \end{aligned}$$

5. (12 pt) Find the area of the region bounded by the curves

$$y = x^2, \quad y = 3x^2, \quad xy = 1, \quad xy = 4\,.$$

Do a change of variables. Set $u = \frac{y}{x^2}$ and $v = xy$. The transformed region is now bounded by the curves

$$u = 1, \quad u = 3, \quad v = 1, \quad v = 4\,.$$

Solving for x and y, we find

$$x = u^{-\frac{1}{3}}v^{\frac{1}{3}}, \quad y = u^{\frac{1}{3}}v^{\frac{2}{3}}\,.$$

Calculating the Jacobian of the transformation, we find that

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}u^{-1}\,.$$

Now,

$$A = \int_1^4 \int_1^3 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^4 \int_1^3 \frac{1}{3}u^{-1}\,du\,dv = \ln(3)\,.$$

6. (12 pt) Evaluate $\displaystyle\int_C yz^2\,ds$ along the helix

$$C:\ \mathbf{r}(t) = 2t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}\,, \quad 0 \le t \le \frac{\pi}{2}\,.$$

$$\begin{aligned}
\int_C yz^2\,ds &= \int_0^{\frac{\pi}{2}} (3\sin t)(3\cos t)^2\sqrt{4 + 9\cos^2 t + 9\sin^2 t}\,dt \\
&= 27\sqrt{13}\int_0^{\frac{\pi}{2}} \sin t\cos^2 t\,dt \qquad \text{(now use } u = \cos t\text{)} \\
&= 27\sqrt{13}\left[-\frac{1}{3}\cos^3 t\right|_0^{\frac{\pi}{2}} \\
&= 9\sqrt{13}
\end{aligned}$$

7. Given the force field $\mathbf{F}(x, y) = e^{2y}\,\mathbf{i} + (1 + 2xe^{2y})\,\mathbf{j}$ and curve

$$C : \mathbf{r}(t) = te^t\,\mathbf{i} + (1 + t)\,\mathbf{j}, \quad 0 \le t \le 1,$$

(a) (12 pt) Find the potential function for $\mathbf{F}$.

Set $f = e^{2y}$ and $g = 1 + 2xe^{2y}$. Since $\frac{\partial f}{\partial y} = 2e^{2y} = \frac{\partial g}{\partial x}$, $\mathbf{F}$ is conservative. We want to find a function ϕ such that $\nabla\phi = \mathbf{F}$. Now,

$$\phi = \int f\,dx = xe^{2y} + k(y)\,.$$

Taking the derviative with respect to y, we find

$$\frac{\partial\phi}{\partial y} = 2xe^{2y} + k'(y)\,.$$

This must be equal to g, so $k'(y) = 1$. Now integrate $k'(y)$ with respect to y to find that $k(y) = y + K$, so

$$\phi = xe^{2y} + y + K\,.$$

Note that any choice for K will make ϕ a potential function for $\mathbf{F}$. In particular, we can choose $K = 0$, so

$$\phi = xe^{2y} + y\,.$$

(b) (12 pt) Find the work done moving a particle along C in the force field $\mathbf{F}$.

Note that the curve starts at $(0, 1)$ and ends at $(e, 2)$, i.e. the points corresponding to $\mathbf{r}(0)$ and $\mathbf{r}(1)$. Using the Fundamental theorem of line integrals, we find that

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(e, 2) - \phi(0, 1) = e^5 + 1\,.$$