Objectives:

- Write Riemann sums efficiently.
- Identify when a sum is an over or underestimate of area.

Motivation: Suppose we wanted to estimate the area between $f(x) = x^2$ and the x-axis from $x = 0$ to $x = 200$ using 100 rectangles. We know how to do this:

$$\text{Area} \approx 0^2 \cdot 2 + 2^2 \cdot 2 + 4^2 \cdot 2 + \ldots + 98^2 \cdot 2$$

However, there must be a shorter way to write this.

Sigma Notation: The symbol $\sum$, pronounced "sigma", stands for "sum".

Examples of sigma notation:

1. Add up all numbers from 1 to 100: $\displaystyle\sum_{i=1}^{100} i$
2. Add up all the squares of nonnegative integers less than 10. $\displaystyle\sum_{i=0}^{9} i^2$
3. Add up five variables labeled $x_3, x_4, \ldots, x_7$. $\displaystyle\sum_{i=3}^{7} x_i$

Using Sigma Notation for Riemann Sums

Suppose we want to come up with an estimate of the area between $f(x)$ and the x-axis between $x = a$ and $x = b$ using n rectangles. We know how to do this using a Riemann sum.

In order to put our sum in Sigma notation, we need to give the rectangle endpoints names.

We'll call the right endpoint of the i^{th} interval x_i. Note that $b = x_n$. Let's give a the name x_0. Notice that the left endpoint of the i^{th} interval is x_{i-1}.

The width of each rectangle is written as Δx. To find the height of the rectangle, we need to choose a point in each smaller interval:

In the i^{th} rectangle, we give this point the name x_i^*. x_i^* can be any x-value in the interval $[x_{i-1}, x_i]$

So our area estimate is given by:

$$\sum_{i=1}^{n} \text{area of } i^{th} \text{ rectangle} = \sum_{i=1}^{n} \text{width} \cdot \text{height} = \sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$$

If we take a right hand sum, $x_i^* =$ ___x_i___ . How can we express x_i^* in terms of known values?

The right endpoint of the first interval is a distance of Δx away from a. We can express this as $x_1 = a + \Delta x$. Then to get the right endpoint of the second interval, we move over by another Δx, so $x_2 = x_1 + \Delta x = a + 2\Delta x$. Moving over another interval width we get $x_3 = a + 3\Delta x$. Continuing in this pattern, $x_i = a + i\Delta x$.

So the value of x_i depends on Δx. How can we express Δx in terms of known values?

$$\Delta x = \frac{b-a}{n}$$

Example Estimate the area between $g(x) = \tan(x^2)$ and the x-axis from $x = -\pi/4$ to $x = \pi/4$ using $n = 10$ intervals.

$\Delta x = \dfrac{\pi/4 - (-\pi/4)}{10} = \dfrac{\pi}{20}$ so $x_i = -\pi/4 + i\Delta x = -\pi/4 + \dfrac{i\pi}{20}$.

So if we use a right hand estimate, our area estimate is given by:

$$\sum_{i=1}^{10} \Delta x g(x_i) = \sum_{i=1}^{10} \left(\frac{\pi}{20}\right) g\left(-\pi/4 + \frac{i\pi}{20}\right) = \sum_{i=1}^{10} \left(\frac{\pi}{20}\right) \tan\left(\left(-\pi/4 + \frac{i\pi}{20}\right)^2\right)$$

If we want to use a left hand estimate, $x_i^* = x_{i-1}$. So the area estimate would be:

$$\sum_{i=1}^{10} \Delta x g(x_{i-1}) = \sum_{i=1}^{10} \frac{\pi}{20} \tan\left(\left(-\pi/4 + \frac{(i-1)\pi}{20}\right)^2\right)$$

We can increase the accuracy of this estimate by ___increasing n___ .

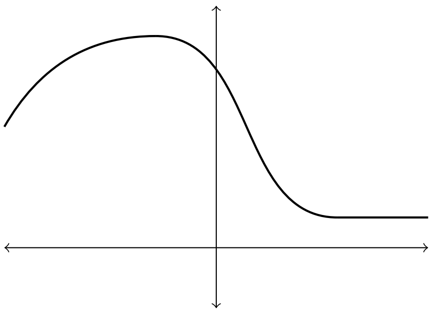
The exact value of the area can be found by ___taking the limit of the estimate as $n \to \infty$___ .

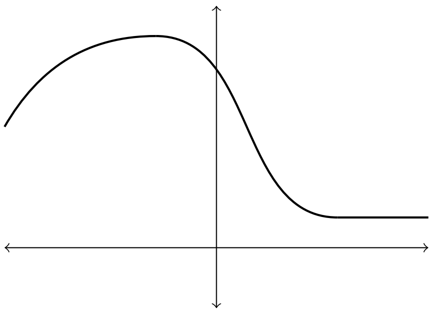
Write an expression for the exact area in this example. (Do not evaluate the limit.)

Area$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x g(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{\pi/4 - (-\pi/4)}{n} g(-\pi/4 + i\Delta x) = \lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{\pi}{2n} g\left(-\pi/4 + \dfrac{i\pi}{2n}\right) =$
$\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{\pi}{2n} \tan\left(\left(-\pi/4 + \dfrac{i\pi}{2n}\right)^2\right)$

Over and Underestimates:

Sketch 5 to 10 rectangles for each estimate. Use your picture to fill in the chart.





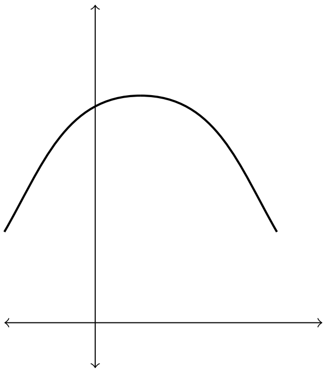
Left Hand Estimate

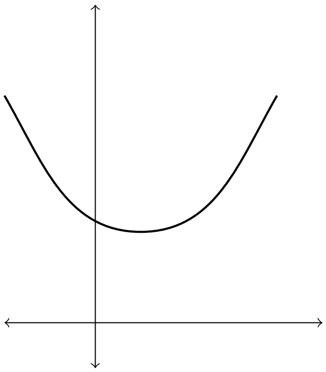
When f is:	LH estimate is:
Constant	exact
Increasing	an underestimate
Decreasing	an overestimate

Right Hand Estimate

When f is:	RH estimate is:
Constant	exact
Increasing	an overestimate
Decreasing	an underestimate

To determine if the **Trapezoidal** rule is an under or overestimate, we need to look at concavity: (These ones will be easier to see if you use very large trapezoids - only 2-3 trapezoids per curve.)





When f is:	Trap estimate is:
Concave down	an underestimate
Concave up	an overestimate