Objectives:

- Prove the Evaluation Theorem
- Practice using the Evaluation Theorem

Intuition: Remember that the antiderivative of velocity is position. Let $s(t)$ be the position in feet of an object at time t in seconds and $v(t)$ be the velocity of the object at time t in feet per second.

Then $s(5) - s(0)$ represents the difference between position at 0 s and position at 5 s.

We could estimate $s(5) - s(0)$ using a Riemann Sum.

E.G. Measure the velocity each second. Add up velocities to get change in distance.
(1 ft/s for 1 s + 2ft/s for 1 s +3ft/s for 1 s + 2ft/s for 1 s −3 ft/s for 1 s = 4ft between position at 0 s and position at 5s)

To get the exact value instead of an estimate, take the limit $n \to \infty$ of the sum with n intervals.

We call this value $\int_0^5 v(t)dt$.

Proof of the Evaluation Theorem:

Let $F(x)$ be any antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$). Note that this means $F(x)$ is differentiable and therefore continuous.

Divide the interval $[a, b]$ into n intervals, with width $\Delta x = \dfrac{b-a}{n}$.

Numberline with $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$

Now, we try to find a way of writing $F(b) - F(a)$ in terms of these points in $[a, b]$:

$$\begin{aligned}
F(b) - F(a) &= F(x_n) - F(x_0) \\
&= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-1}) + \ldots + F(x_2) - F(x_1) + F(x_1) - F(x_0) \\
&= \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right)
\end{aligned}$$

Since $F(x)$ is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i), there exists by the Mean Value Theorem some x_i^* in $[x_{i-1}, x_i]$ such that

$$\begin{aligned}
F'(x_i^*) &= \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \\
f(x_i^*) &= \frac{F(x_i) - F(x_{i-1})}{\Delta x} \\
f(x_i^*)\Delta x &= F(x_i) - F(x_{i-1})
\end{aligned}$$

Returning to our previous equation for $F(b) - F(a)$:

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \\ &= \sum_{i=1}^{n} f(x_i^*)\Delta x \\ \lim_{n\to\infty} (F(b) - F(a)) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\ F(b) - F(a) &= \int_a^b f(x)dx \end{aligned}$$

WOW :D

Using the Evaluation Theorem:
Find $\int_0^\pi -2^x + \sin(x)\, dx$.

$$\begin{aligned} \int_0^\pi -2^x + \sin(x)\, dx &= \left(\frac{-2^x}{\ln(2)} - \cos(x)\right)\Big|_0^\pi = \left(\frac{-2^\pi}{\ln(2)} - \cos(\pi)\right) - \left(\frac{-2^0}{\ln(2)} - \cos(0)\right) \\ &= \frac{-2^\pi}{\ln(2)} + 1 - \left(\frac{-1}{\ln(2)} - 1\right) = \frac{-2^\pi + 1}{\ln(2)} + 2 \approx -9.289 \end{aligned}$$

If $s(x) = x - \sin(x)$ represents the velocity of an object in feet per second, what does the integral above represent?

After π seconds the object is about 9 ft (in the negative direction) away from where it started.

If $f(x) = x - \sin(x)$ represents the increase or decrease in the quantity of fuel carried by a rocket in units of tons per minute, what does the integral above represent?

At π minutes, the rocket has about 9 tons less fuel than at 0 minutes.