Sources.
A Concise Course in Algebraic Topology, J.P. May.
Algebraic Topology, A. Hatcher.

Definition. From the dimension axiom for reduced homology, $\widetilde{H}_0(S^0) \cong \mathbb{Z}$. Fix i_0, a generator of $\widetilde{H}_0(S^0)$. The suspension axiom yields that $\widetilde{H}_{n+1}(S^{n+1}) \xrightarrow{\Sigma} \widetilde{H}_n(S^n) \cong \mathbb{Z}$ is an isomorphism for all $n \geq 0$. Let $i_{n+1} = \Sigma(i_n)$.
For a based topological space X, the *Hurewicz map*, $h : \pi_n(X) \to \widetilde{H}_n(X)$, is given by
$h([f]) = \widetilde{H}_n(f)(i_n)$.

Lemma. *The Hurewicz map is a natural group homomorphism.*

For $[f], [g] \in \pi_n(X)$, $[f] + [g] = [f + g]$ is given by the following composition of maps:

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

where p is the pinch map and ∇ is the fold map.
Consider the composition

$$\widetilde{H}_*(S^n) \xrightarrow{\widetilde{H}_n(p)} \widetilde{H}_n(S^n \vee S^n) \xrightarrow{\widetilde{H}_n(f\vee g)} \widetilde{H}_n(X \vee X) \xrightarrow{\widetilde{H}_n(\nabla)} \widetilde{H}_n(X)$$

The image of i_n under this composite map is $\widetilde{H}_n(\nabla \circ f \vee g \circ p)(i_n) = \widetilde{H}_n(f + g)(i_n) = h([f + g])$.
From the additivity axioms, we have isomorphisms $\widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n \vee S^n)$ and
$\widetilde{H}_n(X) \oplus \widetilde{H}_n(X) \to \widetilde{H}_n(X \vee X)$.

$$\begin{tikzcd}
\widetilde{H}_*(S^n) \arrow[r, "\widetilde{H}_n(p)"] \arrow[dr, dashed] & \widetilde{H}_n(S^n \vee S^n) \arrow[r, "\widetilde{H}_n(f\vee g)"] & \widetilde{H}_n(X \vee X) \arrow[r, "\widetilde{H}_n(\nabla)"] & \widetilde{H}_n(X) \\
& \widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \arrow[u, "\cong"] \arrow[r, dashed] & \widetilde{H}_n(X) \oplus \widetilde{H}_n(X) \arrow[u, "\cong"] \arrow[ur, dashed] &
\end{tikzcd}$$

Let the dashed arrows be the compositions that make the above diagram commute. The image of i_n along the dashed arrows is

$$i_n \mapsto (i_n, i_n) \mapsto (\widetilde{H}_n(f)(i_n), \widetilde{H}_n(g)(i_n)) \mapsto \widetilde{H}_n(f)(i_n) + \widetilde{H}_n(g)(i_n) = h([f]) + h([g])$$

Since the diagram commutes, h is a group homomorphism.
Let $f : X \to Y$ be a map of based spaces.

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\pi_n(f)"] \arrow[d, "h"'] & \pi_n(Y) \arrow[d, "h"] \\
\widetilde{H}_n(X) \arrow[r, "\widetilde{H}_n(f)"'] & \widetilde{H}_n(Y)
\end{tikzcd}$$

Note that $h \circ \pi_n(f)([g]) = h([f \circ g]) = \widetilde{H}_n(f \circ g)(i_n) = \widetilde{H}_n(f) \circ \widetilde{H}_n(g)(i_n) = \widetilde{H}_n(f) \circ h([g])$ for all

$[g] \in \pi_n(X)$. So the above diagram commutes for any f, i.e., h is natural.

Note. The suspension isomorphism Σ is also natural, so $\Sigma \circ h = h \circ \Sigma$.

Lemma. *For any CW-complex X, $\widetilde{H}_n(X) \cong \widetilde{H}_n(X^{n+1})$.*

Let $i \geq n+1$. Since X^i is a subcomplex of X^{i+1}, there exists, by the exactness and suspension axioms for reduced homology, a long exact sequence:

$$\cdots \to \widetilde{H}_{n+1}\big(X^{i+1}/X^i\big) \to \widetilde{H}_n\big(X^i\big) \to \widetilde{H}_n\big(X^{i+1}\big) \to \widetilde{H}_n\big(X^{i+1}/X^i\big) \to \cdots$$

For any i, X^{i+1}/X^i is a wedge of $(i+1)$-spheres. So, $\widetilde{H}_n\Big(X^{i+1}/X^i\Big) \cong \bigoplus_{j\in I} \widetilde{H}_n(S^n_j)$ by additivity. By the suspension axiom, $\widetilde{H}_n(S^{i+1}) \cong \widetilde{H}_{n-1}(S^i) \cong \ldots \cong \widetilde{H}_{n-(i+1)}(S^0)$. Since $n-(i+1) \neq 0$, the dimension axiom yields that $\widetilde{H}_{n-(i+1)}(S^0) \cong 0$. So, $\widetilde{H}_n\Big(X^{i+1}/X^i\Big) \cong \bigoplus_{j\in I} 0$. An identical argument shows $\widetilde{H}_{n+1}\Big(X^{i+1}/X^i\Big) \cong 0$. So, the sequence below is exact:

$$0 \to \widetilde{H}_n\big(X^i\big) \to \widetilde{H}_n\big(X^{i+1}\big) \to 0$$

Thus, $\widetilde{H}_n\big(X^i\big) \cong \widetilde{H}_n\big(X^{i+1}\big)$ for all $i \geq n+1$. As a consequence, $\widetilde{H}_n\big(X^{n+1}\big) \cong \widetilde{H}_n\big(X^j\big)$ for all $j \geq n+1$. So we have colim $\widetilde{H}_n(X^i) \cong \widetilde{H}_n(X^{n+1})$.

In *A Concise Course in Algebraic Topology*, May shows $H_n(X) = \text{colim } H_n(X_i)$ for any $X = X_0 \subseteq X_1 \subseteq \ldots$. The proof uses a construction, tel X_i, formed by attaching the mapping cyclinders for the inclusions $X_i \to X_{i+1}$. This construction is weakly equivalent to X, so $\widetilde{H}_n(\text{tel } X_i) \cong \widetilde{H}_n(X)$. Then the Mayer-Vietoris sequence for particular subspaces of tel X_i and an exact sequence for the colimit of abelian groups are used to show $\widetilde{H}_n(X) \cong \text{colim } \widetilde{H}_n(X_i)$. The proof depends on the additivity and weak equivalence axioms for homology of general topological spaces - for CW-complexes we only need additivity. See pages 114-116 in *Concise* for details.

Therefore, $\widetilde{H}_n(X) \cong \text{colim } \widetilde{H}_n(X^j) \cong \widetilde{H}_n(X^{n+1})$.

Lemma. *Let X be a wedge of n-spheres. Then $h : \pi_n(X) \to \widetilde{H}_n(X)$ is the abelianization homomorphism for $n = 1$ and an isomorphism for $n > 1$.*

If X is a single n-sphere, $\pi_n(X) \cong \mathbb{Z}\{[id]\}$ and $\widetilde{H}_n(X) \cong \mathbb{Z}\{i_n\}$ by the dimension and suspension axioms. Then $h([id]) = \widetilde{H}_n(id)(i_n) = id(i_n) = i_n$, so $\mathbb{Z}\{[id]\} \xrightarrow{h} \mathbb{Z}\{i_n\}$ is an isomorphism. Note that since $\mathbb{Z}\{[id]\}$ is abelian, this also gives the conclusion for $n = 1$.
Now let $X = \bigvee_{j\in I} S^n_j$. By additivity, $\widetilde{H}_n(X) \cong \bigoplus_{j\in I} \widetilde{H}_n(S^n_j) \cong \bigoplus_{j\in I} \mathbb{Z}\{i_n\}$.
For $n > 1$, $\pi_n(X) = \mathbb{Z}\{I\} \cong \underset{j\in I}{\mathbb{Z}}\{[id]\}$. The map h is natural, $\pi_n(X)$ is generated by the inclusions $S^n_j \to X$, and the isomorphism $\bigoplus_{j\in I} \widetilde{H}_n(S^n_j) \to \widetilde{H}_n(X)$ is induced by the inclusions S^n_j, thus the following

diagram commutes:

$$\begin{tikzcd}
\pi_n(S^n_j) \arrow[r] \arrow[d, "h"'] & \pi_n(X) \arrow[d, "h"] \\
\widetilde{H}_n(S^n_j) \arrow[r] & \widetilde{H}_n(X)
\end{tikzcd}$$

In particular, $h(\ldots,0,[id],0,\ldots) = (\ldots,h(0),h([id]),h(0),\ldots) = (\ldots,0,i_n,0,\ldots)$. Then h maps the k-th generator of of $\underset{j\in I}{\mathbb{Z}}\{[id]\}$ to the k-th generator of $\bigoplus\limits_{j\in I}\mathbb{Z}\{i_n\}$ and is therefore an isomorphism.

In the case where $n=1$, $\pi_n(X)\cong F^I$, the free group generated by the inclusions of the n-spheres into X. Then h maps the k-th generator of F^I to the k-th generator of $\bigoplus\limits_{j\in I}\mathbb{Z}\{i_n\} = \left(F^I\right)^{ab}$ and is thus the abelianization homomorphism.