Intro to Homotopy Theory

Katharine Adamyk

August 23, 2017

Outline

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Algebraic Topology

Algebraic Topology

Topological space

Algebraic Topology

Topological space $\Rightarrow$ algebraic structure

Algebraic Topology

Topological space $\Rightarrow$ algebraic structure $\Rightarrow$ info about space

Algebraic Topology

Topological space $\Rightarrow$ algebraic structure $\Rightarrow$ info about space

$\Updownarrow$

All kinds of fun stuff

The World We Live In

Some Preliminaries

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- A based space is a topological space X with a distinguished point $*$.
- A based map is a map of based spaces that takes basepoint to basepoint.

What is a Homotopy?
Formal Definition

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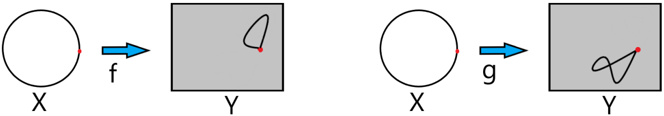
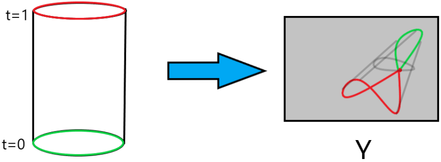
- $H(x, 0) = f(x)$
- $H(x, 1) = g(x)$
- $H(*, t) = *$ for all $t \in I$

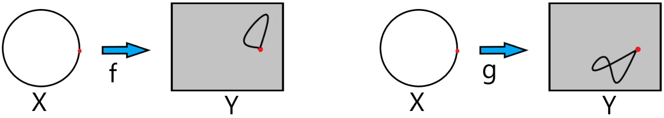
What is a Homotopy?

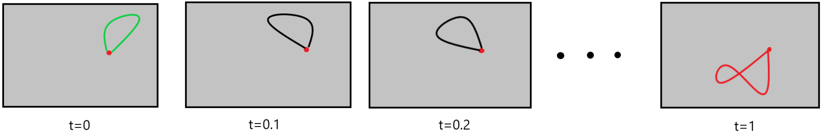
Formal Definition





What is a Homotopy?
Intuition





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So, $\simeq$ is an equivalence relation on maps $X \to Y$.

Outline

- A loop on X is a map $\gamma : S^1 \to X$ where S^1 is the circle.

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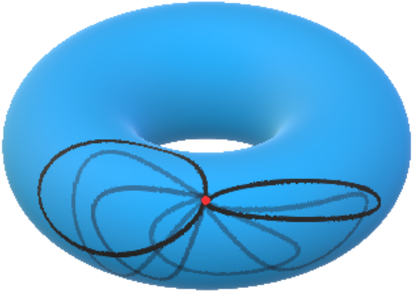
- Two loops are in the same homotopy class if there exists a homotopy between them.



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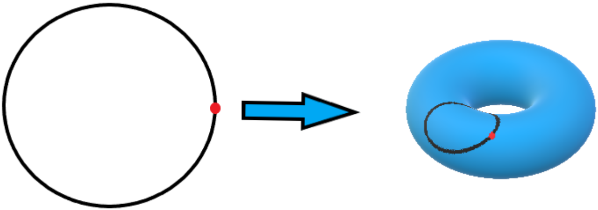
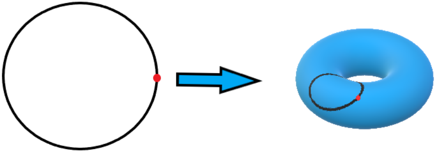
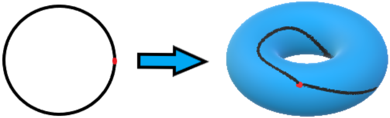


Fundamental Group

Group Operation





$A : S^1 \to X$ $\qquad$ $B : S^1 \to X$

Fundamental Group

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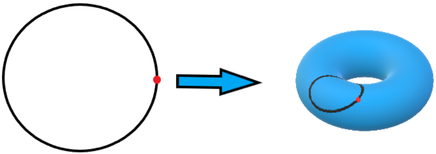

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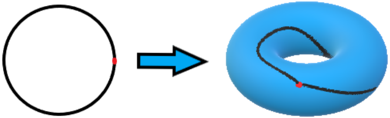
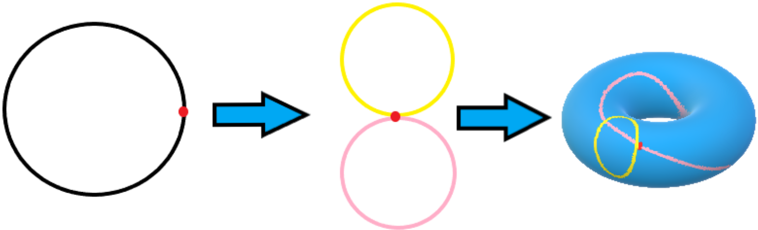


$B : S^1 \to X$



$A + B : S^1 \to X$

Fundamental Group
Identity

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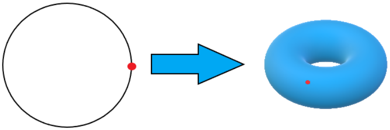

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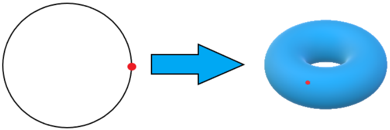

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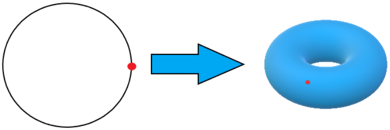

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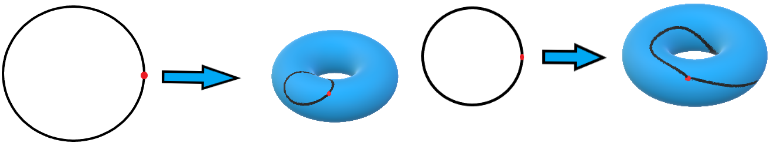

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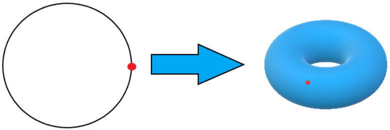

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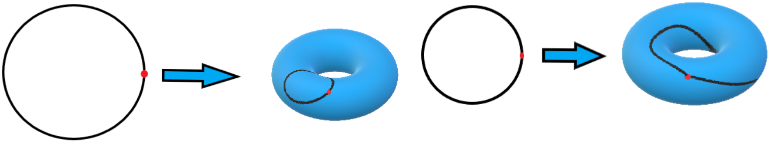

Contractible

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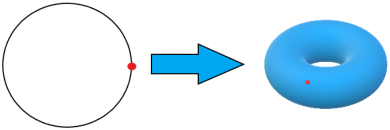

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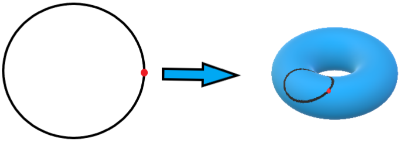
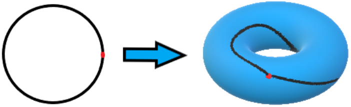




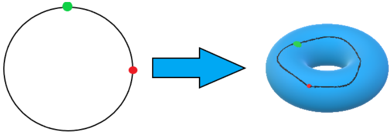
Contractible

Not Contractible

Fundamental Group

Inverses

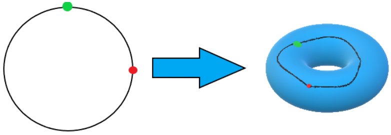
Fundamental Group
Inverses



$$f \in [f]$$

Fundamental Group

Inverses



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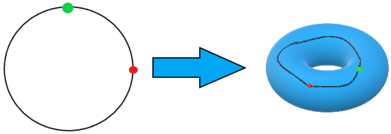
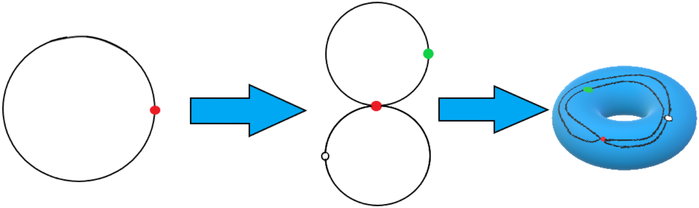
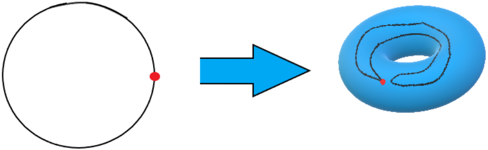


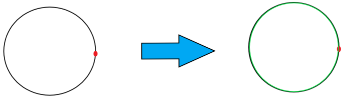
$$g \in [f]^{-1}$$

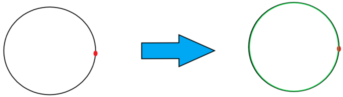


$$\simeq$$



Fundamental Group
Examples





$$\pi_1(S^1) = \mathbb{Z}$$

Fundamental Group

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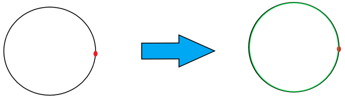

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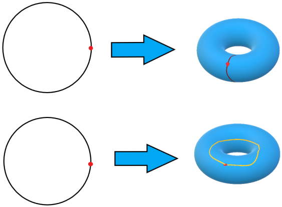

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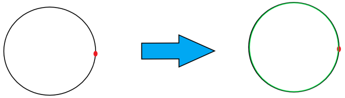

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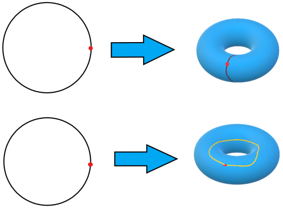

$$\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$$

Fundamental Group

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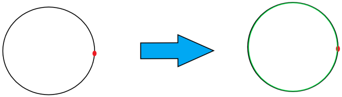

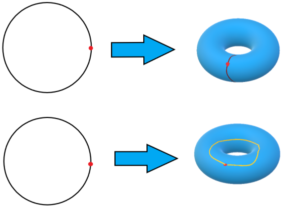

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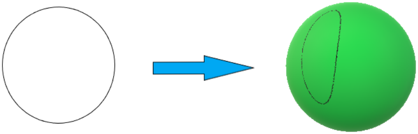

Fundamental Group

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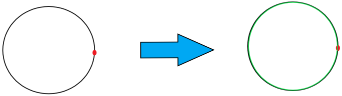

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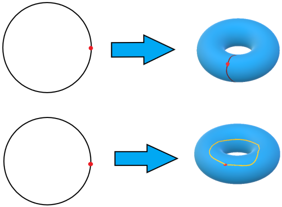

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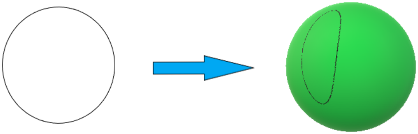

$\pi_1(S) = 0$

Outline

Spheres

Spheres

Definition

The *n-sphere* is

$$S^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sqrt{x_0^2 + x_1^2 + \ldots + x_n^2} = 1 \right\}$$

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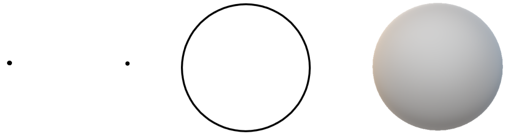

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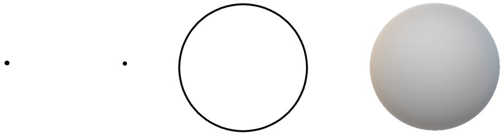

S^0

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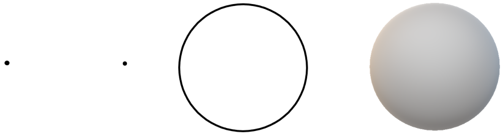

S^0 S^1

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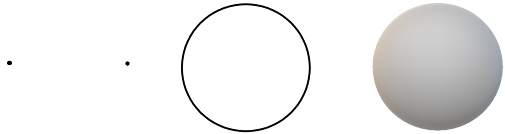

S^0 $\qquad$ S^1 $\qquad$ S^2

Homotopy Group

Definition

The n^{th} homotopy group of a space X, $\pi_n(X)$, is the set of homotopy classes of based maps $S^n \to X$. The group operation is defined by composing the wedge of two maps with the pinch map $S^n \to S^n \vee S^n$.

Outline

The Dream

The Dream

$$\pi_n(S^m)$$

for all $n, m \in \mathbb{Z}_{\geq 0}$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

The Dream

Revised

The Dream
Revised

For large enough k, $\pi_{n+k}(S^k)$ is constant. We call this eventually reached group $\pi_n^S(S^0)$, the n^{th} stable homotopy group of S^0 or the n^{th} stable stem.

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Revised

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$$\pi_n^S(S^0)$$
$$\text{for all } n \in \mathbb{Z}$$

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Localization

One Prime at a Time

With the exception of $\pi_0^S(S^0)$, the stable stems are all finite abelian groups.

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$$\mathbb{Z}/p_1^{e_1} \times \mathbb{Z}/p_2^{e_2} \times \cdots \times \mathbb{Z}/p_m^{e_m}$$

Localization

One Prime at a Time

With the exception of $\pi_0^S(S^0)$, the stable stems are all finite abelian groups. So, they can be decomposed as a product:

$$\mathbb{Z}/p_1^{e_1} \times \mathbb{Z}/p_2^{e_2} \times \cdots \times \mathbb{Z}/p_m^{e_m}$$

One strategy is to see what we can discover about the p-torsion part of $\pi_n(S^0)$, then assemble that information for every prime p.

Thanks!