Localized Ext and Lifting $\mathcal{A}(1)$-Modules

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Plan

The Steenrod Algebra

Cohomology Operations

Throughout this talk, $H^\bullet(—) = \widetilde{H}^\bullet(—; \mathbb{Z}/2)$.

For any $n \in \mathbb{Z}_{\geq 0}$, we have a natural transformation (the n^{th} Steenrod square)

$$Sq^n : H^\bullet(—) \to H^{\bullet+n}(—)$$

that respects suspension

$$\begin{array}{ccc} H^m(X) & \xrightarrow{Sq^n} & H^{m+n}(X) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ H^{m+1}(\Sigma X) & \xrightarrow{Sq^n} & H^{m+n+1}(\Sigma X) \end{array}$$

(and has other nice properties).

The Steenrod Algebra

The Steenrod Algebra

The Steenrod algebra, $\mathcal{A}$, is the graded $\mathbb{Z}/2$-algebra generated by the subset of the Steenrod squares:

$$Sq^{2^n} : H^m(—) \to H^{m+2^n}(—)$$

under the Adem relations:

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

for all $0 < i < 2j$.

For example, $Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$.

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- Sq^1
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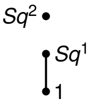

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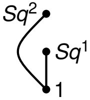

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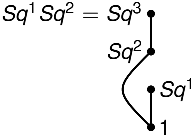

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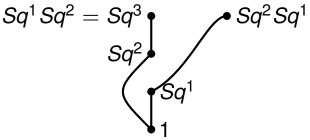

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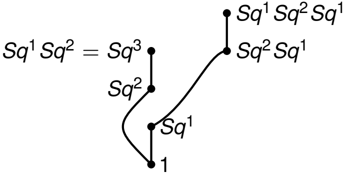

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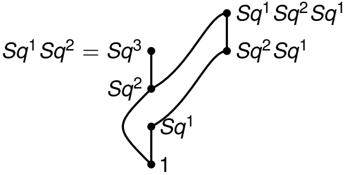

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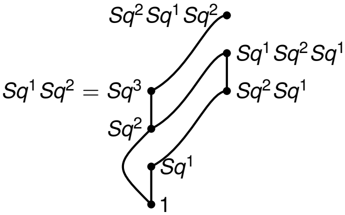

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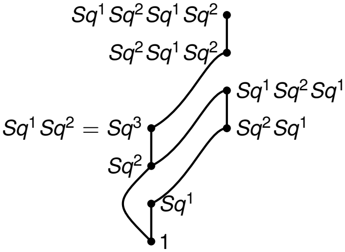

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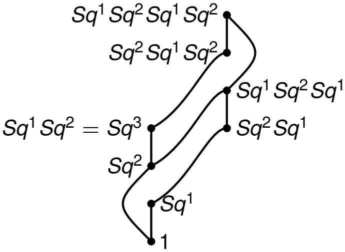

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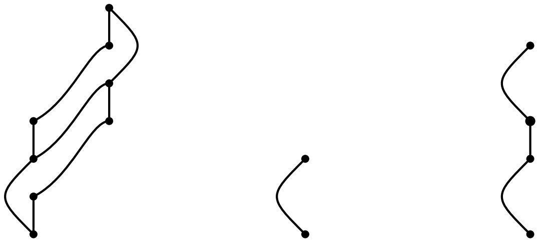
Some $\mathcal{A}(1)$-modules have a compatible $\mathcal{A}$-module structure. Some do not.



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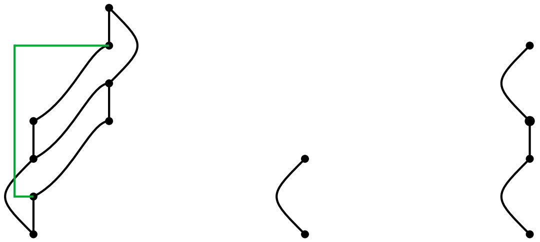

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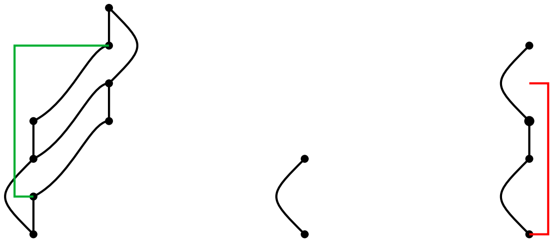

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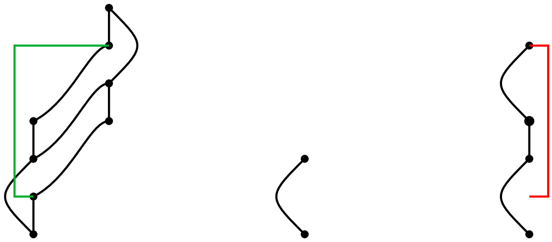

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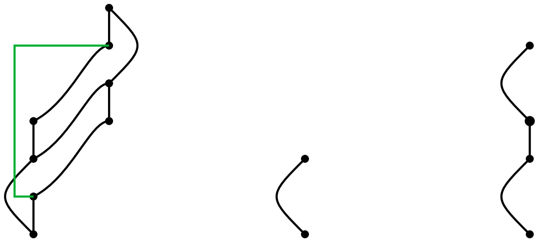


Not an $\mathcal{A}$-module.

$$Sq^2 Sq^1 Sq^2 = Sq^1 Sq^4 + Sq^4 Sq^1$$

Questions

Question 1

Which $\mathcal{A}(1)$-modules can be given a compatible $\mathcal{A}$-module structure?

The Adams Spectral Sequence

The Adams Spectral Sequence

Definition

For a nice space or spectrum X, there exists a spectral sequence

$$E_2^{s,t} = Ext_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{Z}/2) \Rightarrow \pi_{t-s}(X)_2^\wedge$$

Motivating Example

ko

The spectrum *KO* represents real topological K theory. That is,

$$K^n_{\mathbb{R}}(X) = [X, KO]_n$$

There is a spectrum *ko* such that $\pi_n(ko) = \begin{cases} \pi_n(KO) & n \geq 0 \\ 0 & n < 0 \end{cases}$

Thanks to Bott periodicity, we know the homotopy groups of *ko* are 8-periodic:

$$\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, \ldots$$

Motivating Example

The cohomology of connective real K theory is known:

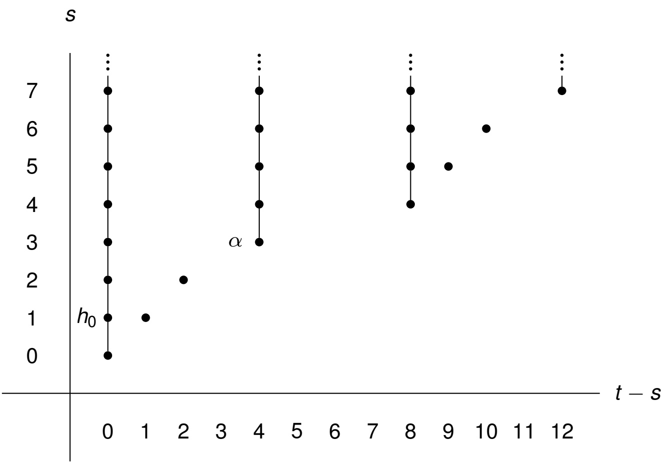
$$H^{\bullet}(ko) = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2 = \mathcal{A}//\mathcal{A}(1)$$

So, the E_2 page of the Adams Spectral Sequence for *ko* is:

$$Ext_{\mathcal{A}}^{s,t}(\mathcal{A}//\mathcal{A}(1), \mathbb{Z}/2) \cong Ext_{\mathcal{A}(1)}^{s,t}(\mathbb{Z}/2, \mathbb{Z}/2)$$

Even better, this spectral sequence collapses on the second page, so we can read off the 2-completed homotopy groups of *ko*.

$Ext^{s,t}_{\mathcal{A}(1)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}(ko)^{\wedge}_2$



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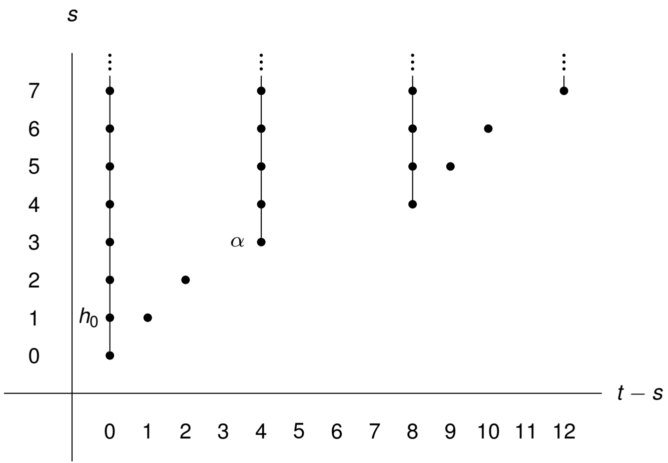
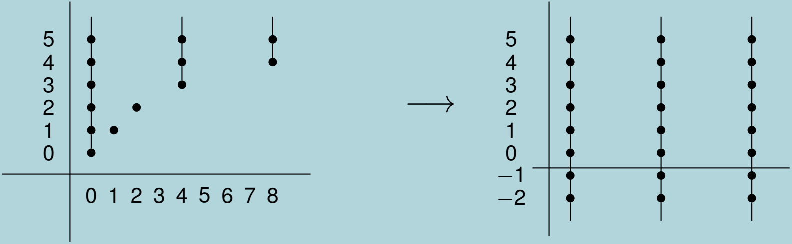


$\pi_n(ko)^{\wedge}_2$ $\mathbb{Z}_2 | \mathbb{Z}/2 | \mathbb{Z}/2 | \ 0 \ | \mathbb{Z}_2 | \ 0 \ | \ 0 \ | \ 0 \ | \mathbb{Z}_2 | \mathbb{Z}/2 | \mathbb{Z}/2 | \ 0 \ | \mathbb{Z}$

Inverting h_0

How do we isolate towers?

Invert h_0.



$Ext_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$ $\qquad$ $h_0^{-1} Ext_{\mathcal{A}(1)}(\mathbb{Z}/2, \mathbb{Z}/2)$

Questions

Question 2

In general, what can we say about $h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$ for an $\mathcal{A}(1)$-module, M?

Margolis Homology

Margolis Homology

In $\mathcal{A}(1)$,

$$Q_0 = Sq^1$$
$$Q_1 = [Sq^{2^1}, Q_0] = Sq^2Sq^1 + Sq^1Sq^2.$$

Importantly, $Q_iQ_i = 0$.

Q_i Margolis Homology

Let M be an $\mathcal{A}(1)$-module.
We have a chain complex,

$$M \xleftarrow{Q_i} M \xleftarrow{Q_i} M$$

The homology of this complex is $H_\bullet(M; Q_i)$, the Q_i Margolis homology of M.

An $\mathcal{A}(1)$-module, M, is called Q_i-local if $H_\bullet(M; Q_j) = 0$ for $j \neq i$.

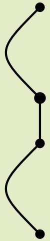
Margolis Homology

Example

$Q_0 = Sq^1$

◯ Kernel of Sq^1

□ Image of Sq^1



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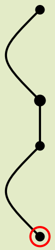

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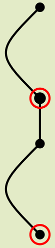

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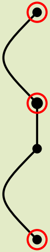

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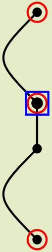

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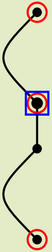

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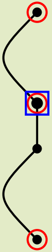

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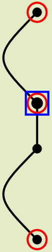

$$H_n(M; Q_0) = \begin{cases} \mathbb{Z}/2 & n = 0, 5 \\ 0 & \text{otherwise} \end{cases}$$

Margolis Homology

Example

$$Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$$

◯ Kernel of Q_1
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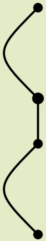

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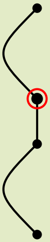

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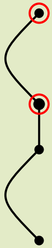

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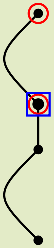

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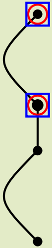

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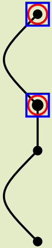


$$H_n(M; Q_1) = 0$$

The Spectral Sequence

Finding h_0 Towers with Margolis Homology

Vanishing Theorem (Adams)

If $H_\bullet(M; Q_0) = 0$ for any $\mathcal{A}(1)$-module M,

$$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2) = 0$$

Finding h_0 Towers with Margolis Homology

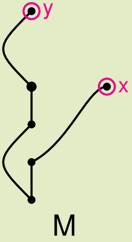
Theorem (Davis)

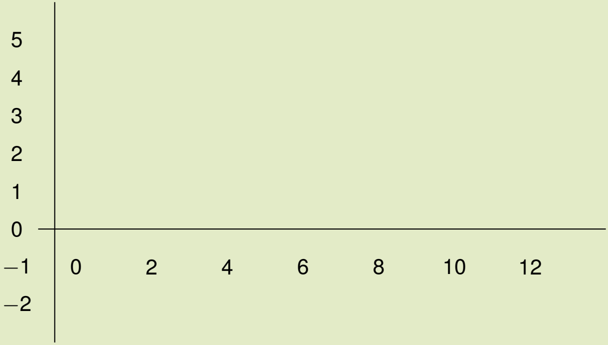
If M is an $\mathcal{A}$-module, then

$$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2) \cong H_\bullet(M; Q_0) \otimes \mathbb{Z}/2[h_0^{\pm 1}, \alpha]$$

$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$

Example





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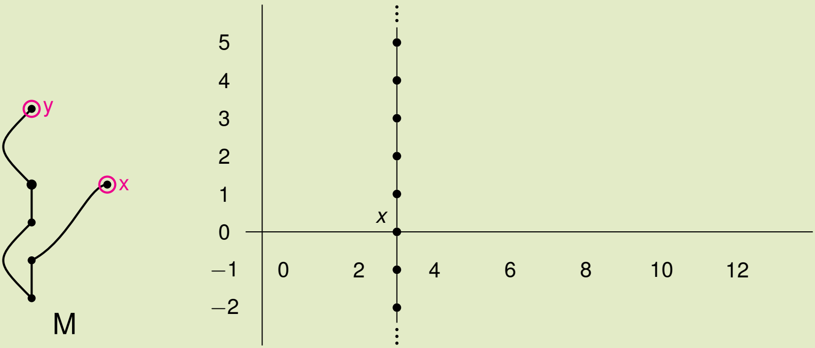

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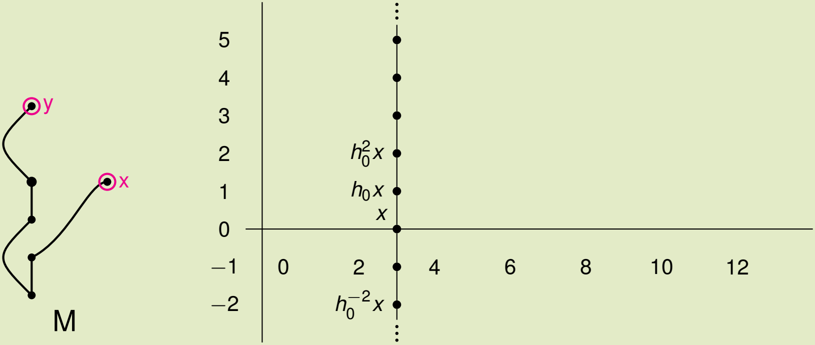

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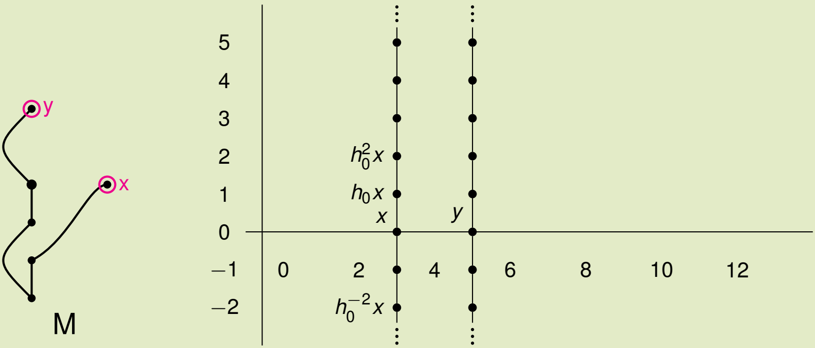

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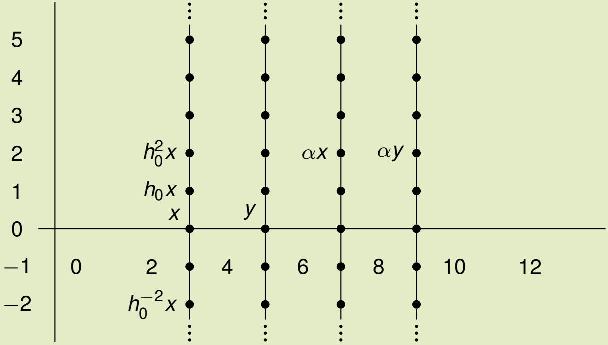
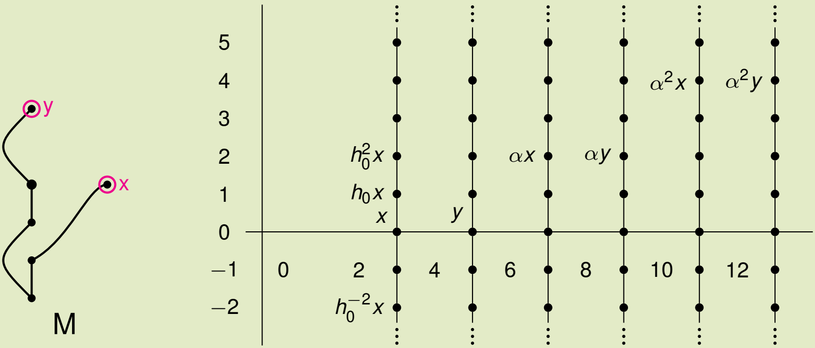
Example



$$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2) \cong H_\bullet(M; Q_0) \otimes \mathbb{Z}/2[h_0^{\pm 1}, \alpha]$$

$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$

Example



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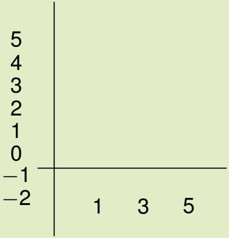
$h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$

However, if M is not an $\mathcal{A}$-module, but merely an $\mathcal{A}(1)$-module, this can fail.

Counterexample

$\mathcal{A}(1)//\mathcal{A}(0)$



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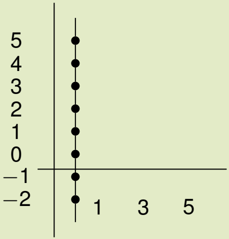


$$h_0^{-1} Ext_{\mathcal{A}(1)}(\mathcal{A}(1)//\mathcal{A}(0), \mathbb{Z}/2) =$$
$$h_0^{-1} Ext_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[h_0^{\pm 1}]$$

Questions

Question 2 (Updated)

Can we describe $h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$ in terms of Q_0-Margolis homology for $\mathcal{A}(1)$-modules that are not necessarily $\mathcal{A}$-modules?

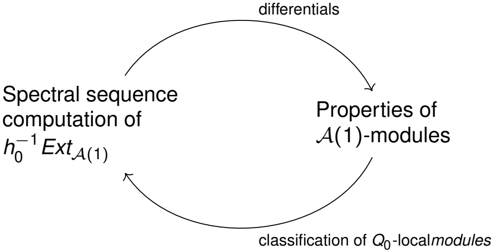
The Spectral Sequence

Proposition (Ricka)

For any bounded below $\mathcal{A}(1)$-module, M, there exists a spectral sequence

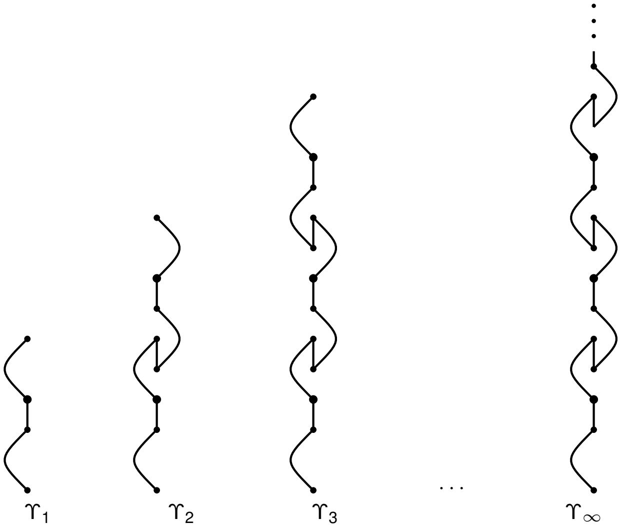
$$E_1 \cong H_\bullet(M; Q_0) \otimes \mathbb{Z}/2[h_0^{\pm 1}, \alpha] \Rightarrow h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{F}_2)$$

If M is an $\mathcal{A}$-module, then this spectral sequence should collapse on the first page. So, any nonzero differentials indicate the lack of a compatible $\mathcal{A}$-module structure.


differentials
Spectral sequence computation of $h_0^{-1} Ext_{\mathcal{A}(1)}$
Properties of $\mathcal{A}(1)$-modules
classification of Q_0-local*modules*

Classification of Q_0-Local $\mathcal{A}(1)$-Modules

The Seagull Modules



The Seagull Modules

Margolis Homology of the Seagulls

For finite n,

$$H_k(\Upsilon_n; Q_0) = \begin{cases} \mathbb{Z}/2 & k = 0, 4n+5 \\ 0 & \text{otherwise} \end{cases}$$

$$H_\bullet(\Upsilon_n; Q_1) = 0$$

When $n = \infty$,

$$H_k(\Upsilon_\infty; Q_0) = \begin{cases} \mathbb{Z}/2 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_\bullet(\Upsilon_\infty; Q_1) = 0$$

(The seagulls are "Q_0-local.")

The Seagull Modules

Proposition (A.)

The spectral sequence

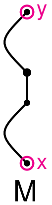
$$E_1 \simeq H_\bullet(\Upsilon_n; Q_0) \otimes \mathbb{Z}/2[h_0^{\pm 1}, \alpha] \Rightarrow h_0^{-1} Ext_{\mathcal{A}(1)}(\Upsilon_n, \mathbb{F}_2)$$

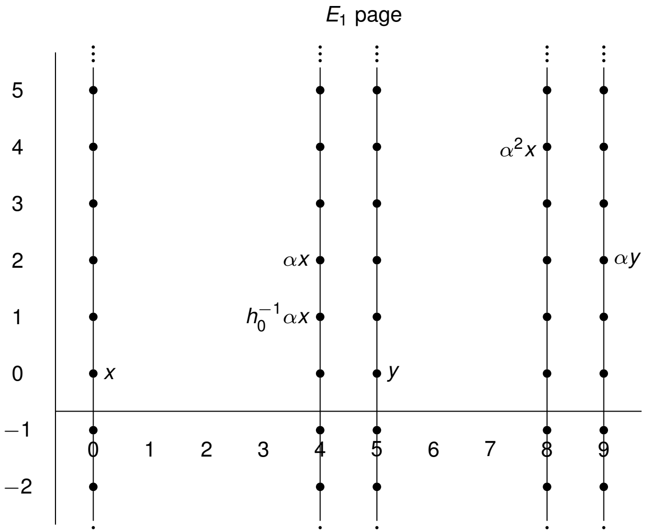
has a nonzero differential d_n (and all other differentials are zero).

Corollary (A.)

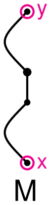
No Υ_n for finite n lifts to an $\mathcal{A}$-module.

Example


y
x
M

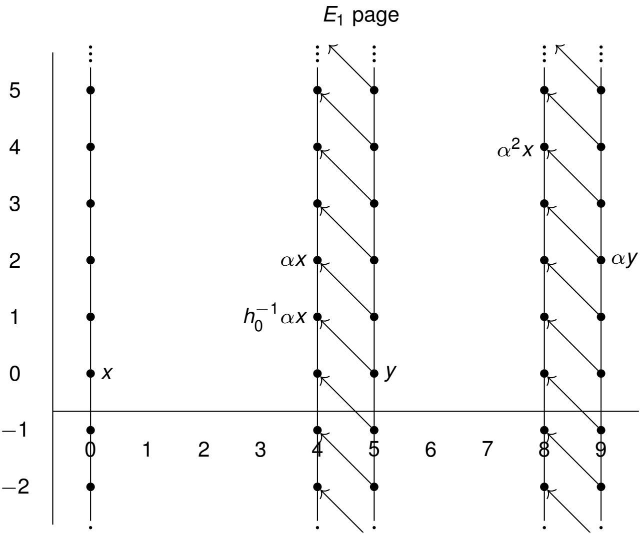

E_1 page
5
4
3
2
1
0
−1
−2
0 1 2 3 4 5 6 7 8 9
x
y
αx
$h_0^{-1}\alpha x$
$\alpha^2 x$
αy

Example

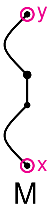


$d_1([x]) = 0$

$d_1([y]) = h_0^{-1}\alpha[x]$



d_1



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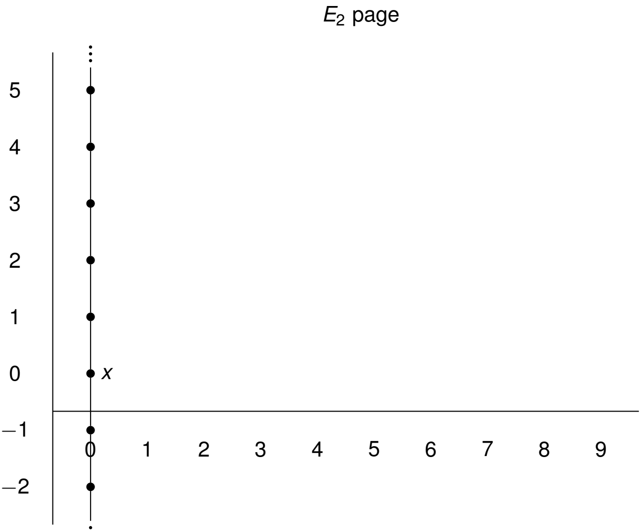
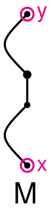


Example



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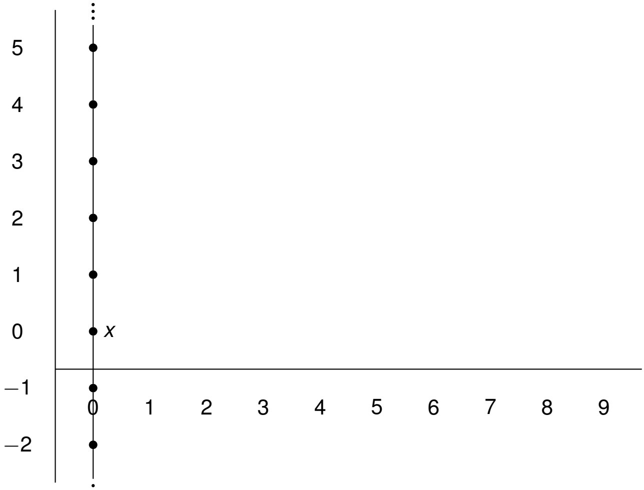
E_∞ page



Classifying $\mathcal{A}(1)$-Modules

Proposition [Conjecture] (A.)

If M is a finite [**bounded below, finite type**] $\mathcal{A}(1)$-module and $H_\bullet(M; Q_1) = 0$, then M is isomorphic to a direct sum of suspensions of Υ_n's.

Probable Fact

If M is a bounded below $\mathcal{A}(1)$-module of finite type, then $H_\bullet(\Upsilon_\infty \otimes M; Q_1) = 0$ and the spectral sequences for M and $\Upsilon_\infty \otimes M$ are isomorphic.

For $n = 1$, we have the correspondence:

Spectral Sequence		$Q_0 -$ local modules		$\mathcal{A}(1)$-modules
There's a nonzero d_1 in $E_1 \Rightarrow h_0^{-1} Ext_{\mathcal{A}(1)}(M, \mathbb{F}_2)$	$\leftrightsquigarrow$	$\Upsilon_\infty \otimes M$ has a direct summand Υ_1	$\leftrightsquigarrow$	M has classes $[x], [y] \in H_\bullet(M; Q_0)$ with $x_* = Sq^2 Sq^1 Sq^2 y_*$

Goal

Determine the condition for an $\mathcal{A}(1)$-module, M that corresponds to Υ_n appearing as a summand of $\Upsilon_\infty \otimes M$.

Thank you!

Questions?