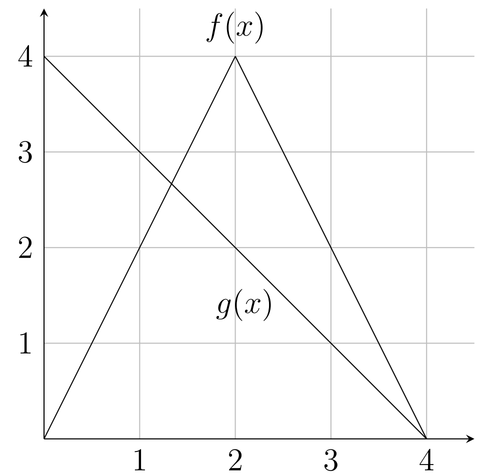
Objectives:

- Apply our new analytic derivative techniques to previous problems: graphing, tangent lines, and real world applications.

Use the graph of functions $f(x)$ and $g(x)$ to answer the following questions.



1. Let $h(x) = f(x) \cdot g(x)$. Find $h'(1)$.

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

$$\text{so } h'(1) = f'(1)g(1) + g'(1)f(1) = f'(1)(3) + g'(1)(2) = (2)(3) + (-1)(2) = 6 - 2 = 4$$

2. Let $h(x) = f(x) \cdot g(x)$. Find $h'(2)$.

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

but $f'(2)$ is undefined. To be absolutely sure that $f'(2)g(2) + g'(2)f(2)$ as a whole is undefined, we could take left and right limits.

3. Let $\ell(x) = f(g(x))$. Find $\ell'(3)$.

$$\ell'(x) = f'(g(x))g'(x) \text{ so } \ell'(3) = f'(g(3))g'(3) = f'(1)g'(1) = (2)(-1) = -2$$

4. Let $k(x) = f(x)/g(x)$. Find $k'(3)$.

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\text{so } k'(3) = \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} = \frac{(1)f'(3) - (2)g'(3)}{(1)^2} = f'(3) - 3g'(3) = -2 - 3(-1) = 2$$

Graphing f Using Derivatives

Remember that we can estimate a graph of f given where f' and f'' are positive or negative.

Given a formula for f, we can calculate f' and f'' and find where each one is positive or negative.

Example:

Graph $f(x) = -x^3 + 6x^2 - 9x$ using $f'(x)$ and $f''(x)$.

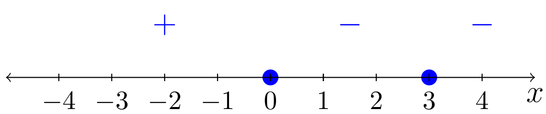
Step 1: Where is $f(x)$ positive or negative?

(a) Solve for where $f(x) = 0$:

$$f(x) = -x^3 + 6x^2 - 9x = -x(x^2 - 6x^2 + 9) = -x(x-3)(x-3) \qquad x = 0, 3$$

(b) Mark the points where $f(x) = 0$ on the real number line into intervals. Test a point in each interval and mark if $f(x)$ is positive or negative on that interval.

Number line for $f(x)$:



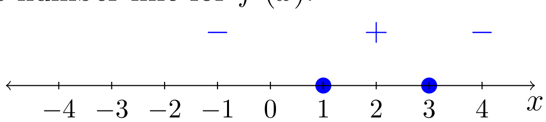
Step 2: Where is $f'(x)$ positive or negative?

(a) Find $f'(x)$ and solve for where $f'(x) = 0$:

$$f'(x) = -3x^2 + 12x - 9 = -3\left(x^2 - 4x + 3\right) = -3(x-1)(x-3) \qquad x = 1, 3$$

(b) Test points to label the number line for $f'(x)$:

Number line for $f'(x)$:



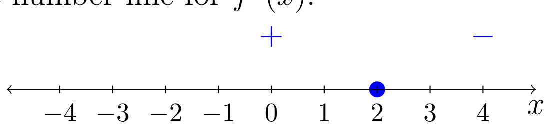
Step 3: Where is $f''(x)$ positive or negative?

(a) Find $f''(x)$ and solve for where $f''(x) = 0$:

$$f''(x) = -6x + 12 \qquad x = 2$$

(b) Test points to label the number line for $f''(x)$:

Number line for $f''(x)$:

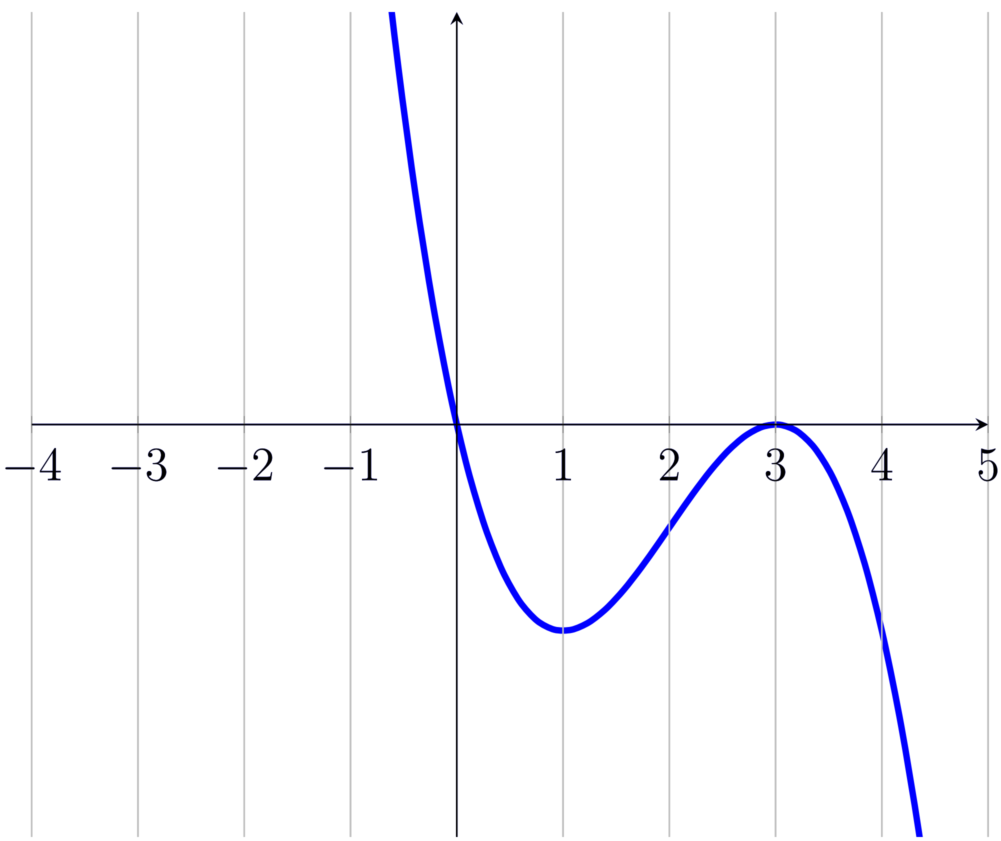


Step 4: Use all the points where $f(x) = 0$, $f'(x) = 0$ or $f''(x) = 0$ to divide the real number line into intervals. Using the previous number lines, note for each interval whether $f(x)$ is positive/negative, increasing/decreasing and concave up/down.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
$f(x)$ negative or positive	positive	negative	negative	negative	negative
$f(x)$ increasing or decreasing	decreasing	decreasing	increasing	increasing	decreasing
$f(x)$ concave up or down	up	up	up	down	down

Step 5: We are finally ready to graph $f(x)$!

Don't forget that we already know $f(x) = 0$ for $x =$ 0, 3



Finding Tangent Lines

Find the tangent line to $f(x) = \sin(x)\tan(x)$ at $x = \frac{\pi}{4}$.

Note: It's always worth seeing if we can simplify before taking the derivative. $\sin(x)\tan(x) = \frac{\sin^2(x)}{\cos(x)}$, so that's not any nicer. I'd rather use the product rule than the quotient rule, so let's stick with $\sin(x)\tan(x)$.

$$f'(x) = \sin(x)\sec^2(x) + \cos(x)\tan(x) = \frac{\sin(x)}{\cos^2(x)} + \cos(x)\frac{\sin(x)}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)} + \sin(x)$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos^2\left(\frac{\pi}{4}\right)} + \left(\frac{\pi}{4}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)} + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{\frac{1}{2}}\right) + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

So, the slope of the tangent line is $m = \frac{3}{\sqrt{2}}$. The other thing we need is the point on the curve at $x = \frac{\pi}{2}$.

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)\left(\frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}\right) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \left(\frac{1}{\sqrt{2}}\right)$$

So, the tangent line to $f(x)$ at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is:

$$y - \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

Interpreting the Derivative

You are applying to receive land from a buffalo farming cooperative. Let b represent the number of buffalo you have in your herd. The cooperative will grant you a plot of land that has width $1.5^{1/b}$ km and length $b^{1/2}$ km.

(a) Write an equation for the area of land you are given if you have b buffalo. By careful to identify units.

$$A(b) = 1.5^{1/b}b^{1/2}$$

The units for b are number of buffalo, the units of $A(b)$ are km^2.

(b) Find the derivative of your function at $b = 100$. Include units. Explain what this derivative represents.

$$A'(b) = \ln(1.5)1.5^{1/b}(-b^{-2})(b^{1/2}) + 1.5^{1/b}\left(\frac{1}{2}b^{-1/2}\right) = \frac{-\ln(1.5)1.5^{1/b}}{b^{3/2}} + \frac{1.5^{1/b}}{2b^{1/2}}$$

$$A'(100) \approx 0.0498\frac{\text{km}^2}{\text{buffalo}}$$

So if you have 100 buffalo, adding one additional buffalo to the herd increases the amount of land you're entitled to by 0.0498 km^2 (or 49,800 m^2 if you prefer).

More Derivative Practice:

Find the derivatives of the following functions.

a) $f(x) = (x^2 - \sqrt{x})3^x$

$f'(x) = (x^2 - \sqrt{x})3^x \ln(3) + 3^x \left(2x - \frac{1}{2\sqrt{x}}\right)$

b) $h(y) = 4^y(\pi^3 - y^2)$

$h'(y) = (\pi^3 - y^2)4^y(\ln(y)) + 4^y(-2y)$

c) $f(\theta) = \big(\sin(\theta)\big)^2 \cot(\theta)\csc(\theta)$ [***Hint:*** Rewrite $f(\theta)$ in terms of sine and cosine.]

$f(x) = \sin(\theta)\sin(\theta)\left(\frac{\cos(\theta)}{\sin(\theta)}\right)\left(\frac{1}{\sin(\theta)}\right) = \cos(\theta)$ so $f'(x) = \sin(\theta)$

d) $y = \frac{t+1}{2^t}$

$y' = \frac{2^t - (t+1)2^t \log 2}{2^{2t}}$

e) $y = 2^{\cot(x)}$

$y' = -\ln(2)2^{\cot(x)}(\csc(x))^2$

f) $f(z) = \dfrac{z^2+1}{\sqrt{z}}$

$f'(z) = \dfrac{3z^2-1}{2z^{(3/2)}}$

g) $f(x) = \sin(\tan(x))$

$f'(x) = \cos(\tan(x))\big(\sec(x)\big)^2$

h) $h(t) = \dfrac{1+\cos(t)}{t+\sin(t)}$

$h'(t)) = \dfrac{(x+\sin(t))(-\sin(t)-(1+\cos(t))(1+\cos(t))}{(x+\sin(t))^2}$

i) $f(x) = \dfrac{\tan(x)-1}{\sec(x)}$

$f'(x) = \sin(x) + \cos(x)$

j) $y = \sqrt{9x^2+1}$

$y' = \dfrac{18x}{2\sqrt{9x^2+1}}$