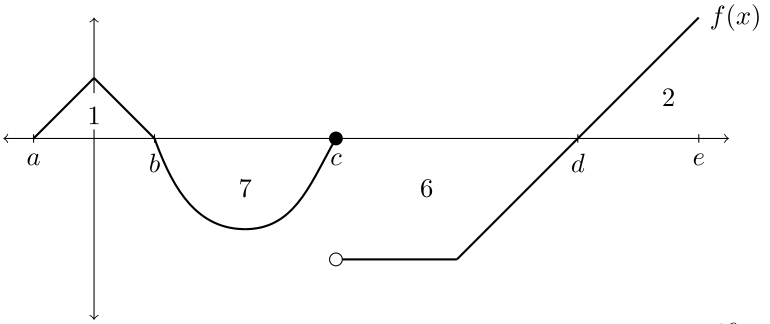
Objectives:

- Find the integral of the absolute value of a given function.
- Interpret the integral of a function's absolute value.

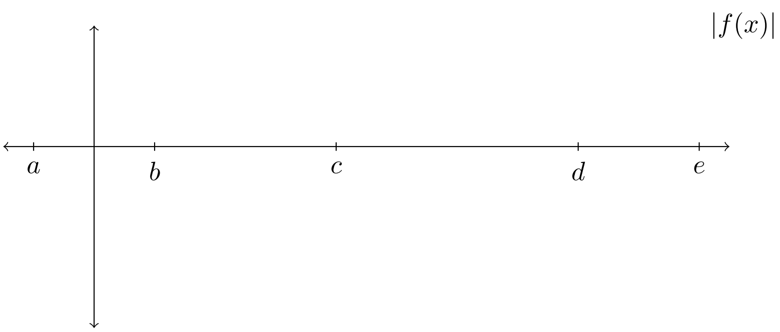
Graphically:



The graph of $f(x)$ is given with areas labeled (e.g. $\int_b^c f(x)dx = 7$).

$$\int_a^e f(x)dx = 1-7-6+2 = -10$$

Graph $|f(x)|$:



$$\int_a^e |f(x)|dx = 1+7+6+2=16$$

Write $\int_a^e |f(x)|dx$ in terms of integrals of $f(x)$:

$$\int_a^e |f(x)|dx = \int_a^b f(x)dx - \int_b^c f(x)dx - \int_c^d f(x)dx + \int_d^e f(x)dx$$

So to find $\int_a^b |f(x)|dx$,

1. Find all points where $f(x) =$ __0__
2. Use these points to separate $[a, b]$ into subintervals where $f(x)$ __doesn't change sign__ .
3. To calculate $\int_a^b |f(x)|dx$, add the absolute value of the integral of each subinterval. (Add the integral if $f(x) \geq 0$ on that subinterval and subtract the integral if $f(x) \leq 0$.)

Example: $\displaystyle\int_{-\pi}^{\pi} |\cos(x)|dx$

(Draw a picture.)
$\cos(x) = 0$ at $x = -\pi/2, \pi/2$
$\cos(x) < 0$ for $[-\pi, -\pi/2)$ and $(\pi/2, \pi]$ and $\cos(x) > 0$ for $[-\pi/2, \pi/2]$. So,

$$\begin{aligned}\int_{-\pi}^{\pi} |\cos(x)|dx &= -\int_{-\pi}^{-\pi/2} \cos(x)dx + \int_{-\pi/2}^{\pi/2} \cos(x)dx - \int_{\pi/2}^{\pi} \cos(x)dx \\ &= (\sin(-\pi/2) - \sin(-\pi)) + (\sin(\pi/2) - \sin(\pi/2)) - (\sin(\pi) - \sin(\pi/2)) \\ &= -(-1-0) + (1-(-1)) - (0-1) = 1 + 2 + 1 = 4\end{aligned}$$

Interpreting the Integral of Absolute Value

If $v(t)$ represents an object's velocity as a function of time, we know the integral of $v(t)$ represents __displacement, the net change in position__ .

The integral of speed, $|v(t)|$, represents __total distance traveled__ .

Example: The velocity of a boat in miles per hour at x hours after noon is given by $v(x) = x^2 - 4x + 3$.

(a) What is the total distance traveled by the boat between noon and 4pm?
Hint: $x^2 - 4x + 3 = (x-3)(x-1)$

$$\begin{aligned}\int_0^4 |v(x)|\ dx &= \int_0^1 x^2 - 4x + 3\ dx - \int_1^3 x^2 - 4x + 3\ dx + \int_3^4 x^2 - 4x + 3\ dx \\ &= \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_0^1 - \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_1^3 + \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_3^4 \\ &= \frac{7}{3} - \left(-\frac{7}{3}\right) + \frac{4}{3} = \frac{18}{3} = 6 \text{ miles}\end{aligned}$$

(b) What is the distance between the boat's starting position at noon and the boat's ending position at 4pm?

$$\int_0^4 v(x)\ dx = \int_0^4 x^2 - 4x + 3\ dx = \left(\frac{x^3}{3} - 2x^2 + 3x\right)\Big|_0^4 = \frac{4}{3}$$