Objectives:

- Review the chain rule for derivatives.
- Define and use u-substitution for integrals.

Review: Compute the following derivatives:

1. $h(x) = f(g(x))$

   $h'(x) = f'(g(x))g'(x)$

2. $f(x) = (x^3 + 2x)^5$

   $f'(x) = 5\left(x^3 + 2x\right)^4 (3x + 2)$

3. $g(x) = e^{x^2}$

   $g'(x) = e^{x^2}(2x)$

4. $h(t) = (\sin(t))^3$

   $h'(t) = 3\left(\sin(t)\right)^2 \cos(t)$

5. $k(x) = \ln(\tan(x))$

   $k'(x) = \frac{1}{\tan(x)} \left(\sec(x)\right)^2$

6. $\ell(w) = \arctan(\cos(w))$

   $\ell'(w) = \frac{1}{1 + \left(\cos(x)\right)^2}(-\sin(x))$

Substitution:

So, we now know $\int \frac{(\sec(x))^2}{\tan(x)} dx =$ $\underline{\quad \ln(\tan(x)) + C \quad}$.

In general, $\int f'(g(x))g'(x)\, dx =$ $\underline{\quad f(g(x)) + C \quad}$.

Another way we can write this is $\int f'(u)\, du =$ $\underline{\quad f(u) + C \quad}$. We call this substituting u for $g(x)$ and often refer to this method as ___"u-substitution"___.

Examples:

1. $\int e^{\sin(x)} \cos(x)\, dx$

Pick $u = \sin(x)$ so then $du = \cos(x)dx$.

$$\int e^u\, du = e^u + C = e^{\sin(x)} + C$$

We can check by taking the derivative: $\frac{d}{dx} e^{\sin(x)} = e^{\sin(x)} \cos(x)$

2. $\int 3x^2 \left(x^3 + 5\right)^{10}\, dx$

Pick $u = x^3 + 5$ so then $du = 3x^2\, dx$.

$$\int u^{10}\, du = \frac{u^{11}}{11} + C = \frac{(x^3+5)^{11}}{11} + C$$

(Notice that not every choice for u is equally helpful. For example, if we picked $u = x^3$ we would've been stuck with the integral $\int (u+5)^{10}\, du$, which is not as nice as $\int u^{10}\, du$ at all. In general, we might have to do some experimentation to see which choice of u [if any] simplifies the integral the best.)

3. $\int \frac{1}{t+2} dt$

$u = t + 2$, $du = (1)dt$

$$\int \frac{1}{u}\, du = \ln(u) + C = \ln(t+2) + C$$

Sometimes we have to manipulate the integral before using substitution:

Example: $\int x^2\left(x^3+7\right)^5\ dx$

Method 1: Solve for dx:

If we want $u = x^3+7$ then $du = 3x^2dx$ so $\frac{1}{3x^2}du = dx$. So,

$$\int x^2\left(x^3+7\right)^5 dx = \int x^2u^5\left(\frac{1}{3x^2}du\right) = \int \frac{1}{3}u^5du = \frac{1}{3}\cdot\frac{u^6}{6} = \frac{\left(x^3+7\right)^6}{18}$$

Method 2: "Fix-it-up":

We want $u = x^3+7$ so we'd really like to see $3x^2dx$ in our integral. Massage the integral so it looks like this:

$$\int x^2\left(x^3+7\right)^5\ dx = \int \frac{1}{3}\cdot 3x^2\left(x^3+7\right)^5\ dx = \int \frac{1}{3}u^6\ du$$

Then proceed in the same way as in method 1.

Definite Integrals Using u-Substitution:

Example: $\int_0^1 x^2\left(1+2x^3\right)^4 dx$

Method 1 : Change limits of integration:

We pick $u = 1+2x^3$, $du = 6x^2$. Then if $x=0$, $u = 1+2(0)^3 = 1$ and if $x = 1$, $u = 1+2(1)^3 = 3$. So we have:

$$\int_0^1 x^2\left(1+2x^3\right)^4 dx = \int_1^3 \frac{1}{6}u^4du = \frac{1}{6}\cdot\frac{u^5}{5}\Big|_1^3 = \frac{1}{6}\cdot\frac{3^5}{5} - \frac{1}{6}\cdot\frac{1^5}{5} = \frac{243}{30} - \frac{1}{30} = \frac{242}{30}$$

Method 2 : Find integral in terms of original variable, then substitute.

Use same u, du as above but keep the x limits of integration:

$$\int_0^1 x^2\left(1+2x^3\right)^4 dx = \int_{x=0}^{x=1} \frac{1}{6}u^4du = \frac{1}{6}\cdot\frac{u^5}{5}\Big|_{x=0}^{x=1} = \frac{1}{6}\cdot\frac{\left(1+2x^3\right)^5}{5}\Big|_0^1 = \frac{(1+2(1)^3)^5}{30} - \frac{(1+2(0)^3)^5}{30} = \frac{242}{30}$$