Objectives:

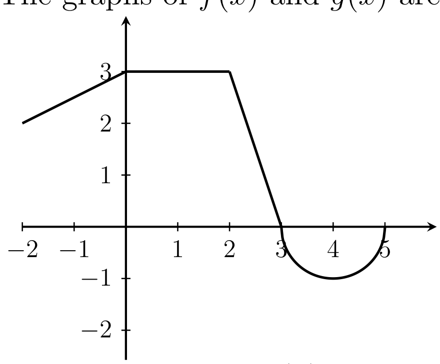
- Practice using properties of definite integrals.
- Compare values of definite integrals.
- Use antiderivatives to evaluate definite integrals.

1. $\int_0^{2\pi} (x + \sin(x))\, dx$

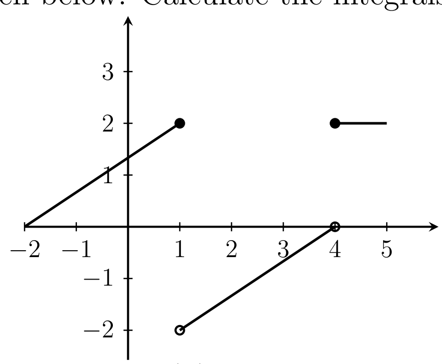
$$\int_0^{2\pi} x dx = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(0 + i\frac{2\pi}{n}\right)\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4\pi^2}{n^2} i = \lim_{n\to\infty} \frac{4\pi^2}{n^2} \sum_{i=1}^{n} i$$
$$= \lim_{n\to\infty} \frac{4\pi^2}{n}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} 4\pi\frac{n+1}{n} = \lim_{n\to\infty} 4\pi\frac{1}{1} = 4\pi$$

Then draw $\sin(x)$ on $[0, 2\pi]$ to see that the area is 0. So,
$\int_0^{2\pi} (x + \sin(x))\, dx = \int_0^{2\pi} x dx + \int_0^{2\pi} \sin(x) dx = 4\pi + 0 = 4\pi$

2. The graphs of $f(x)$ and $g(x)$ are given below. Calculate the integrals.



$f(x)$



$g(x)$

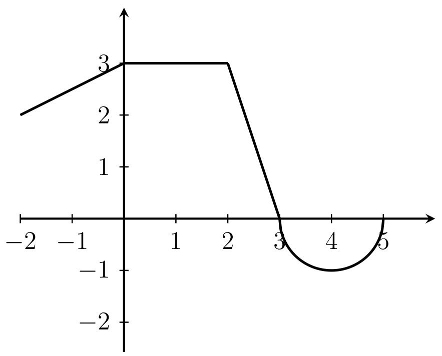
(a) $\int_{-2}^{1} f(x) + g(x) dx = \int_{-2}^{1} f(x)dx + \int_{-2}^{1} g(x)dx = \left(\left(\frac{2+3}{2}\right)2 + 3(1)\right) + \frac{1}{2}(3)(2) = 11$

(b) $\int_2^5 10 f(x)dx = 10\int_{-2}^{1} f(x)dx = 10\left(\frac{1}{2}(3)(1) + \frac{1}{2}\pi(1)^2\right) = 15 - 5\pi$

(c) $\int_{-2}^{1} g(x) + 5dx = \int_{-2}^{1} g(x)dx + \int_{-2}^{1} 5dx = \frac{1}{2}(3)(2) + 5\big(1 - (-2)\big) = 18$

(d) $\int_2^1 g(x)dx + \int_4^2 g(x)dx = \int_4^1 g(x)dx = -\int_1^4 g(x)dx = -\left(\frac{1}{2}(3)(-2)\right) = -(-3) = 3$

Comparing Integrals: For the function $f(x)$ in the previous problem, draw a function $h(x)$ on the axis such that $h(x) \geq f(x)$ for all x values in the interval $[0, 5]$:



How does $\int_0^5 h(x)$ compare to $\int_0^5 f(x)$?

In general we can say that if $f(x)$ ___$\leq$___ $h(x)$ for all x in the interval $[a, b]$, then $\int_a^b f(x)$ ___$\leq$___ $\int_a^b h(x)$.

In particular:

(1) If $f(x) \geq 0$ for all x in $[a, b]$:

$\int_a^b f(x)dx \geq 0$

(2) If $m \leq f(x) \leq M$ for all x in $[a, b]$ where m, M are constants:

$m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

Example: It would be very difficult to calculate $\int_{-2}^{3} \sin\left(\frac{1}{x}\right) dx$. However, we can compare the integral we want to know about to integrals that are easy to compute:

Would probably be useful to draw the function as a reminder of why it's so difficult to integrate.

We know $-1 \leq \sin(x) \leq 1$. So, $\int_{-2}^{3} -1dx \leq \int_{-2}^{3} \sin\left(\frac{1}{x}\right) dx \leq \int_{-2}^{3} 1dx$.
From our rectangle knowledge and/or rule from last time, we can see,

$$-1(3-(-2)) \leq \int_{-2}^{3} \sin\left(\frac{1}{x}\right) dx \leq 1(3-(-2))$$
$$-5 \leq \int_{-2}^{3} \sin\left(\frac{1}{x}\right) dx \leq 5$$

Evaluation Theorem (or, Fundamental Theorem of Calculus, Part II)

If f is ___continuous___ on ___the closed interval $[a, b]$___

and F is any ___antiderivative of f___, (i.e. $F'(x) =$ ___$f(x)$___), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

We use the notation ___$F(x)\Big|_a^b$___ to denote ___$F(b) - F(a)$___.

Note:

Hey!! What happens to the $+C$??

The most general antiderivative of $f(x)$ is $F(x) + C$, so why don't we evaluate $(F(x) + C)\Big|_a^b$ instead? If we do, we get $(F(b) + C) - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$. We are not going to leave $+C$ out of antiderivatives in general, this is a special case.

Examples

1. $\int_{-1}^{2} x^4 dx$

   $\int_5^8 x^4 dx = \frac{x^5}{5}\Big|_{-1}^{2} = \frac{2^5}{5} - \frac{(-1)^5}{5} = \frac{32}{5} + \frac{1}{5} = \frac{33}{5}$

2. $\int_0^1 \frac{1}{1+x^2} dx$

   $\int_0^1 \frac{1}{1+x^2} dx = (\arctan(x))\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0$

3. $\int_2^{10} \left(e^x + 5x - \frac{1}{x}\right) dx$

   $\int_2^{10} \left(e^x + 5x - \frac{1}{x}\right) dx = \left(e^x + \frac{5}{2}x^2 - ln|x|\right)\Big|_2^{10} = \left(e^{10} + \frac{5}{2}(100) - \ln|100|\right) - \left(e^2 + \frac{5}{2}(4) - \ln|4|\right)$

Because of this relationship between the integral of $f(x)$ and the antiderivative of $f(x)$, we write $\int f(x)dx$ to mean the antiderivative of $f(x)$. We call this expression an indefinite integral.

Note: Key Difference: A definite integral is a number. An indefinite integral is a function. Note that the theorem above says that if $\int f(x)dx = F(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

So now we have 3 ways of calculating an indefinite integral:
(1) a graphical area (2) a limit of Riemann sums and (3) using antiderivatives.

Interpreting the integral:
The Evaluation Theorem also appears as the Net Change Theorem.

Since $F'(x) = f(x)$ is the rate of change of $F(x)$, the Evaluation Theorem tell us that the integral from a to b of the rate of change, $F'(x)$ is equal to $F(b) - F(a)$, which we call the "net change" of $F(x)$ from a to b.

Examples:

If $f(x)$ represents:	Then $\int_a^b f(x)dx = F(b) - F(a)$ represents:
Velocity	Postion(b) - Position (a) ="displacement" over $[a, b]$
Marginal Cost	Cost(b) - Cost(a) = net change in cost over $[a, b]$
Growth Rate of a Population	Population(b) - Population(x) = net change in population

Note: NET change only measures difference between the value at a and the value at b.
E.G., if you leave home at $t = 0$ hr, drive 10 miles and then get back home at $t = .5$ hr, $\int_a^b s(t)dt = 0$, NOT 10 miles. We'll talk about how to compute the total distance travled later.